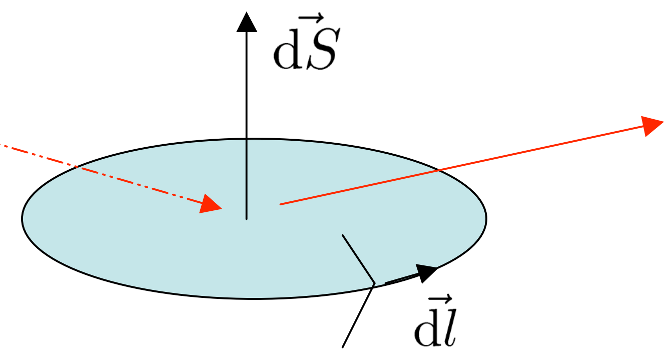
Lecture 10: Curl

- Consider any vector field $\vec{A}$
- And a loop in one plane with vector area $\mathrm{d}\vec{S}$
- If $\vec{A}$ is ***conservative*** then

$$\oint \vec{A}.\mathrm{d}\vec{l} = 0 \text{ and } \vec{A} = \vec{\nabla}\phi$$



- But in general $\oint \vec{A}.\mathrm{d}\vec{l} \neq 0$
- And we introduce a new (***pseudo***) vector
- Of magnitude:

$$\mathrm{curl}\vec{A} \qquad \lim_{\mathrm{area}\to 0} \frac{\oint \vec{A}.\mathrm{d}\vec{l}}{\mathrm{area}}$$

Can evaluate 3 components by taking areas with normals in *xyz* directions

- Which depends on the orientation of the loop
- The direction normal to the direction of the plane (of area $=|\mathrm{d}\vec{S}|$) which maximises the line integral

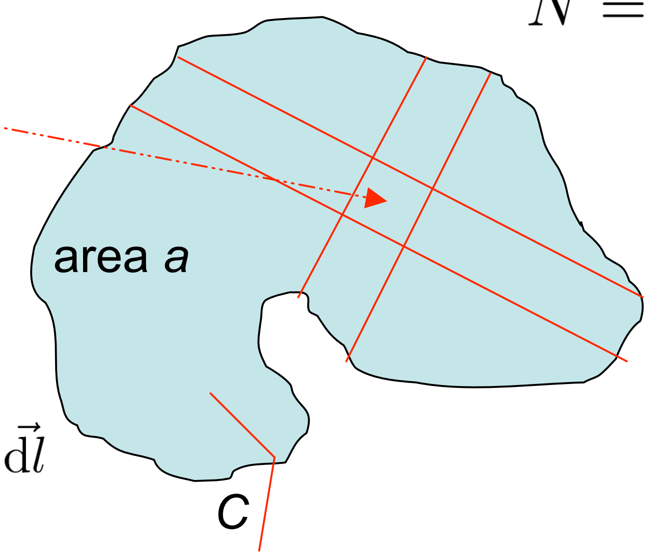
$$\oint \vec{A}.\mathrm{d}\vec{l} = I_{\max}$$

...is the direction of $\mathrm{curl}\vec{A}$ (will prove soon)

Curl is independent of (small) loop shape

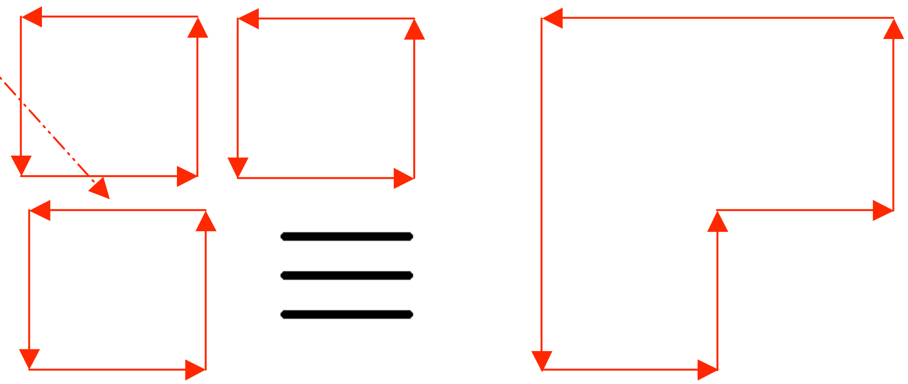
$$N = \frac{a}{a_{\mathrm{rec}}}$$

- Sub-divide loop into N tiny rectangles of area a_{rec}
- Summing line integrals, the interior contributions cancel, yielding a line integral over boundary C



$$\frac{1}{a}\oint_C \vec{A}.\mathrm{d}\vec{l} = \sum_N \frac{1}{a}\oint_{\mathrm{rec}} \vec{A}.\mathrm{d}\vec{l} = N\frac{a_{\mathrm{rec}}}{a}\frac{1}{a_{\mathrm{rec}}}\oint \vec{A}.\mathrm{d}\vec{l}$$

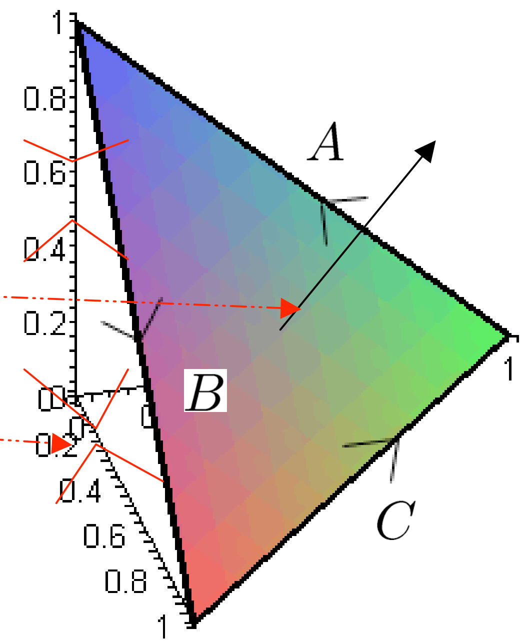
- Thus $\oint \vec{A}.\mathrm{d}\vec{l}/\mathrm{area}$ is same for large loop as for each rectangle
- Hence, curl $\vec{A}$ is independent of loop shape
- But loop does need itself to be vanishingly small



Curl is a Vector

- Imagine a loop of arbitrary direction
- Make it out of three loops with normals in the $\vec{i}, \vec{j}, \vec{k}$ directions

$$\mathrm{d}\vec{S} = (\mathrm{d}S_x, \mathrm{d}S_y, \mathrm{d}S_z)$$



- Because interior loops cancel, we get

$$\oint_{ABC} \vec{A}.\mathrm{d}\vec{l} = \oint_{A(-\vec{k})(\vec{j})} + \oint_{B(-\vec{i})(\vec{k})} + \oint_{C(-\vec{j})(\vec{i})} = C_x\mathrm{d}S_x + C_y\mathrm{d}S_y + C_z\mathrm{d}S_z$$

$$\text{where } \mathrm{curl}\vec{A} = (C_x, C_y, C_z)$$

$$\Rightarrow \oint \vec{A}.\mathrm{d}\vec{l} \equiv (\mathrm{curl}\vec{A}).\mathrm{d}\vec{S}$$

- Explains why line integral = $I_{\max}\cos\theta$

  where θ is angle between loop normal and direction of curl $\vec{A}$

In Cartesian Coordinates

- Consider tiny rectangle in *yz* plane

$$\oint \vec{A}.\mathrm{d}\vec{l} \text{ with } \vec{A} = (A_x, A_y, A_z)$$
$$\mathrm{d}\vec{l} = (0, \mathrm{d}y, \mathrm{d}z)$$

- Contributions from sides 1 and 2

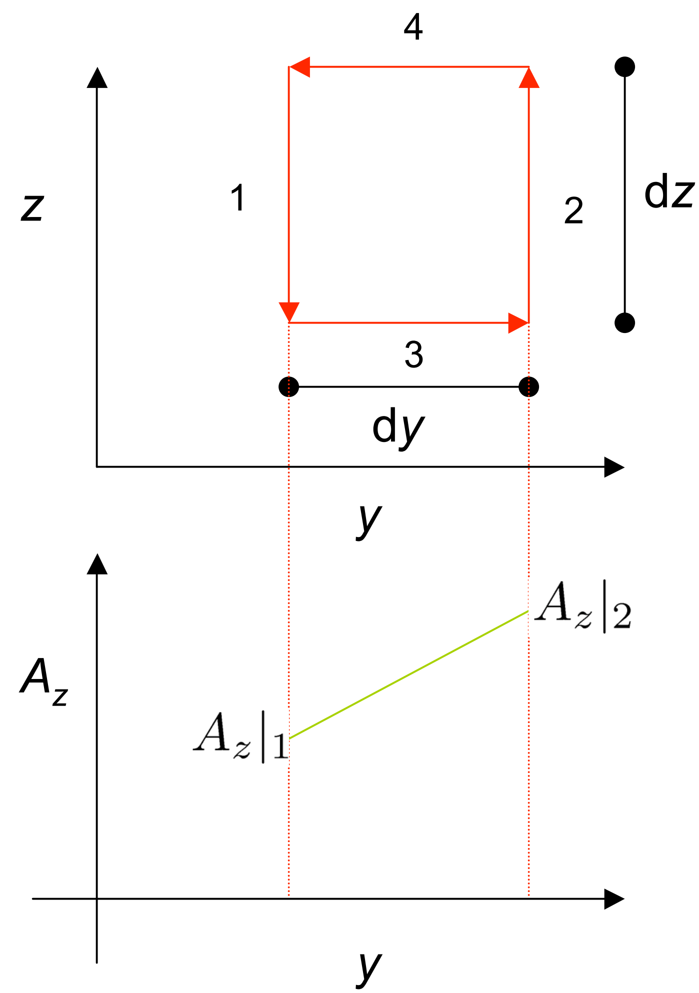
$$-(A_z|1)\mathrm{d}z + (A_z|2)\mathrm{d}z = \mathrm{d}z\,[A_z|2 - A_z|1] =$$
$$\mathrm{d}z\left[\mathrm{d}y\frac{\partial A_z}{\partial y}\right]$$

- Contributions from sides 3 and 4

$$-(A_y|4)\mathrm{d}y + (A_y|3)\mathrm{d}y = \mathrm{d}y\,[A_y|3 - A_y|4] =$$
$$-\,\mathrm{d}y\left[\mathrm{d}z\frac{\partial A_y}{\partial z}\right]$$

- Yielding in total

$$\oint_{yz} \vec{A}.\mathrm{d}\vec{l} = \left[\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right]\mathrm{d}y\mathrm{d}z$$



• Repeating this exercise for a rectangular loop in the *zx* plane yields

$$\oint \vec{A}.\mathrm{d}\vec{l} = \left[\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right] \mathrm{d}x\mathrm{d}z$$

• and in the *xy* plane yields

$$\oint \vec{A}.\mathrm{d}\vec{l} = \left[\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right] \mathrm{d}x\mathrm{d}y$$

•

$$\oint \vec{A}.\mathrm{d}\vec{l} \equiv (\mathrm{curl}\vec{A}).\mathrm{d}\vec{S} = (C_x, C_y, C_z).\mathrm{d}\vec{S}$$

$$\text{where } (C_x, C_y, C_z) = \left[\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}, \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}, \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right]$$

Key Equations

$$\oint \vec{A}.\mathrm{d}\vec{l} \equiv (\mathrm{curl}\vec{A}).\mathrm{d}\vec{S}$$

$$\mathrm{curl}\vec{A} = \left[\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}, \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}, \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right]$$

$$\mathrm{curl}\vec{A} = \vec{\nabla} \times \vec{A} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

Simple Examples

$$\vec{\nabla} \times \vec{r} = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}) \times (x, y, z)$$

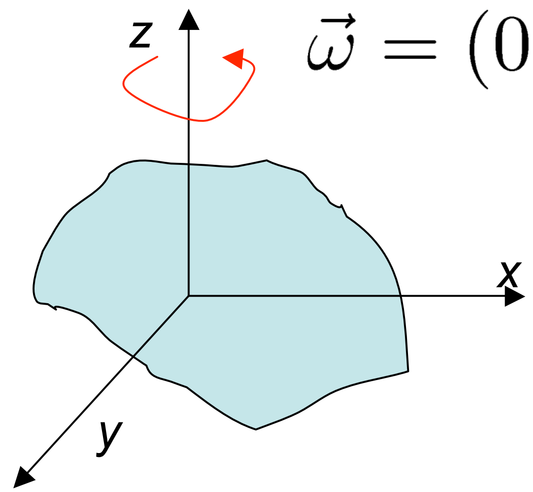
$$= (\frac{\partial z}{\partial y} - \frac{\partial y}{\partial z}, \frac{\partial x}{\partial z} - \frac{\partial z}{\partial x}, \frac{\partial y}{\partial x} - \frac{\partial x}{\partial y}) = (0, 0, 0)$$

$$\vec{\nabla} \times (-y, x, z)$$

$$= (\frac{\partial z}{\partial y} - \frac{\partial x}{\partial z}, \frac{\partial(-y)}{\partial z} - \frac{\partial z}{\partial x}, \frac{\partial x}{\partial x} - \frac{\partial(-y)}{\partial y}) = (0, 0, 2)$$

Curl and Rotation

- Consider a ***solid body*** rotating with angular velocity ω about the *z* axis



$$\vec{v} = \vec{\omega} \times \vec{r}$$

- describes the velocity vector field
- Evaluate curl $\vec{v}$ from the definition
- *x*-component:

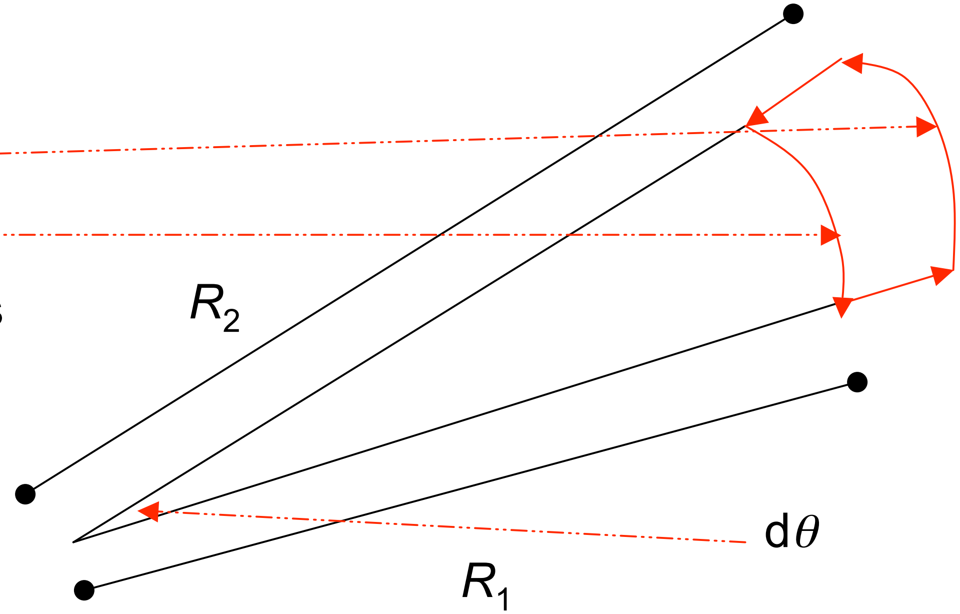
$$\oint \vec{v}.\mathrm{d}\vec{l} \text{ in } yz \text{ plane} = 0 \text{ as } \vec{v} \text{ is perpendicular to } \mathrm{d}\vec{l}$$

- Similarly for *y*-component

$$\oint \vec{v}.\mathrm{d}\vec{l} \text{ in } xz \text{ plane} = 0 \text{ as } \vec{v} \text{ is perpendicular to } \mathrm{d}\vec{l}$$

- As only the *z* component is non-zero we see that $\vec{\omega}$ and curl $\vec{v}$ will be parallel

- *z* component
- In a rotating fluid, its easier to
  \`swim round' here than
  here
- i.e. ***local evidence*** that the fluid is
  rotating



$$[\mathrm{curl}\ \vec{v}]_k \equiv \frac{(R_2\omega)R_2\mathrm{d}\theta - (R_1\omega)R_1\mathrm{d}\theta}{\frac{1}{2}(R_2^2 - R_1^2)\mathrm{d}\theta} = 2\omega$$

- So magnitude of curl is proportional to any macroscopic rotation

$\mathrm{If\ curl}\ \vec{A} = 0 \Rightarrow$ Flow is said to be ***irrotational***

- In a more general case, curl is a function of position