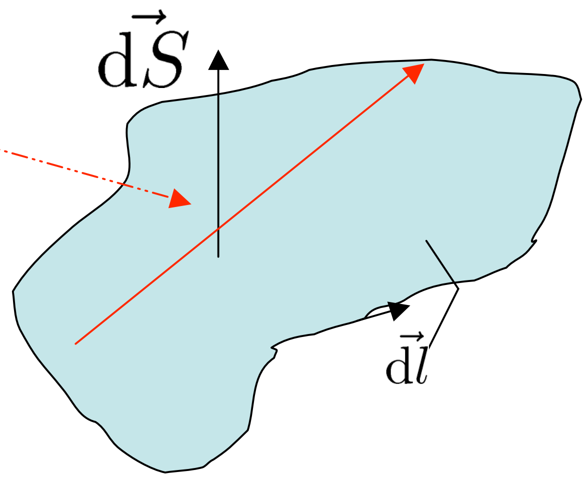
Lecture 11: Stokes Theorem

- Consider a surface *S*, embedded in a vector field $\vec{A}$
- Assume it is bounded by a rim (not necessarily planar)
- For each small loop

$$\oint \vec{A}.\mathrm{d}\vec{l} = \mathrm{d}\vec{S}.(\vec{\nabla} \times \vec{A})$$

- For whole loop (given that all interior boundaries cancel in the normal way)



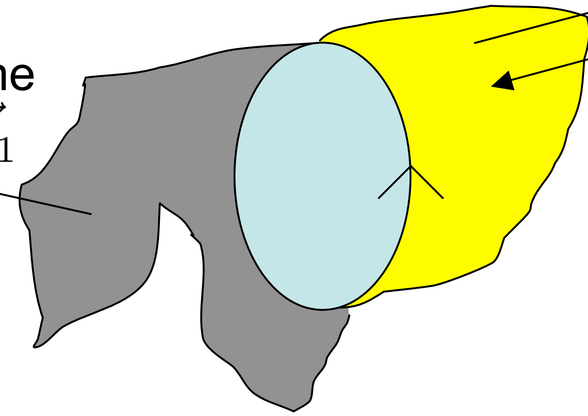
$$\oint \vec{A}.\mathrm{d}\vec{l} = \int\int_S \mathrm{d}\vec{S}.(\vec{\nabla} \times \vec{A})$$

OUTER RIM SURFACE INTEGRAL OVER ***ANY***

SURFACE WHICH SPANS RIM

Consequences

- Curl $\vec{A}$ is ***incompressible***
- Consider two surfaces (S_1 + S_2) with same rim $\int\int(\vec{\nabla}\times\vec{A}).\mathrm{d}\vec{S}$ must be the the same for both surfaces (by ***Stokes Theorem***)



- For ***closed*** surface formed by $S_1 + S_2'$

$$\Rightarrow \int\int \mathrm{d}\vec{S}.(\vec{\nabla}\times\vec{A}) = 0$$

$$\Rightarrow \mathrm{div}(\mathrm{curl}\vec{A}) = 0$$ using the ***divergence theorem***

- Also for a conservative field

$$\oint \vec{A}\cdot\mathrm{d}\vec{l} = 0 \Rightarrow \vec{A} = \vec{\nabla}\phi \overset{\text{by Stokes Theorem}}{\Rightarrow} \mathrm{curl}\vec{A} = \vec{0} \Rightarrow \mathrm{curl}(\mathrm{grad}\phi) = \vec{0}$$

CONSERVATIVE = IRROTATIONAL

- These are examples (see Lecture 12) of ***second-order differential operators*** in vector calculus

Example

$$\vec{A} = (xy^2 + z, x^2y + 2, x) \Rightarrow \vec{\nabla} \times \vec{A} = (0, 1 - 1, 2xy - 2xy) = \vec{0}$$

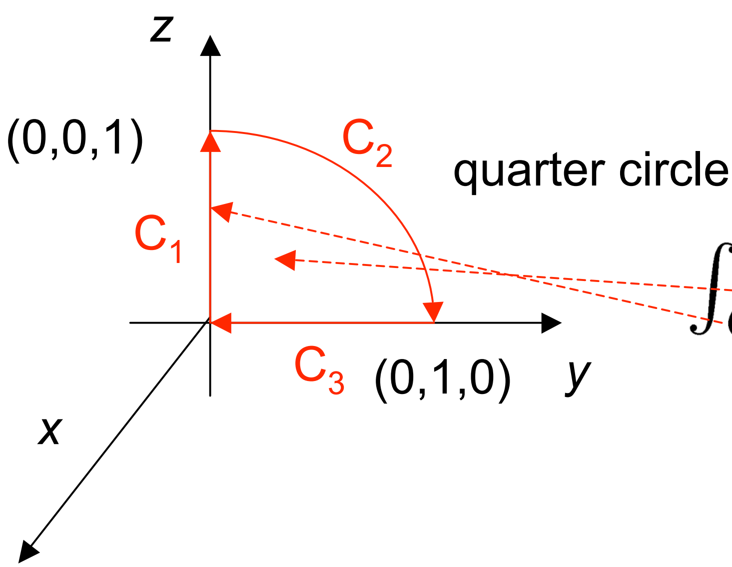
- So this is a ***conservative*** field, so we should be able to find a ***potential*** ϕ

$$\frac{\partial \phi}{\partial x} = xy^2 + z \Rightarrow \phi = \frac{1}{2}x^2y^2 + zx + f(y, z)$$
$$\frac{\partial \phi}{\partial y} = x^2y + 2 \Rightarrow \phi = \frac{1}{2}x^2y^2 + 2y + g(x, z)$$
$$\frac{\partial \phi}{\partial z} = x \Rightarrow \phi = xz + h(x, y)$$

- All can be made consistent if

$$\phi = \frac{1}{2}x^2y^2 + zx + 2y + k \text{ where } k \text{ is a constant}$$

Another Example



$$\vec{F}(x,y,z) = (y,z,x)$$

$$\int_C \vec{F}\cdot \mathrm{d}\vec{l} = \int\int_S (\vec{\nabla}\times\vec{F})\cdot \mathrm{d}\vec{S}$$

$$\vec{\nabla}\times\vec{F} = (-1,-1,-1)$$

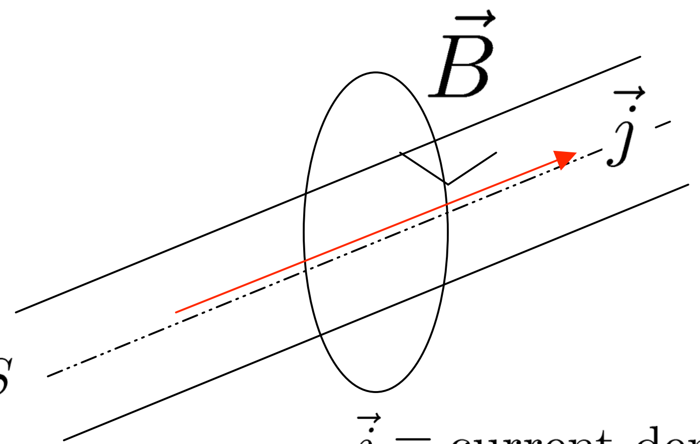
$$\mathrm{d}\vec{S} = (0,-1,0)\mathrm{d}y\mathrm{d}z$$

$$\Rightarrow \int_C \vec{F}\cdot \mathrm{d}\vec{l} = \frac{\pi}{4}$$

- Check via direct integration $\int_C = \int_{C_1} + \int_{C_2} + \int_{C_3} = 0 + \frac{\pi}{4} + 0$

Physical Example

- Magnetic Field $\vec{B}$ due to a steady current *I*
- ***"Ampere's Law"***



$$\oint \vec{B}.\mathrm{d}\vec{l} = \mu_0 \times (\text{current enclosed}) = \mu_0 \int\int \vec{j}.\mathrm{d}\vec{S}$$

$\vec{j} \equiv$ current density

applying Stoke's Theorem

$$\Rightarrow \int\int \mathrm{d}\vec{S}.\vec{\nabla} \times \vec{B} = \mu_0 \int\int \vec{j}.\mathrm{d}\vec{S}$$

units: A m^{-2}

$$\Rightarrow \mathrm{curl}\vec{B} = \vec{\nabla} \times \vec{B} = \mu_0 \vec{j}$$

Differential form of Ampere's Law

- Since $\mathrm{div}(\mathrm{curl}\vec{B}) = 0 \Rightarrow \mathrm{div}\vec{j} = 0$
- $\vec{j}$ obeys a ***continuity equation*** with no ***sources*** or ***sinks*** (see Lecture 13)

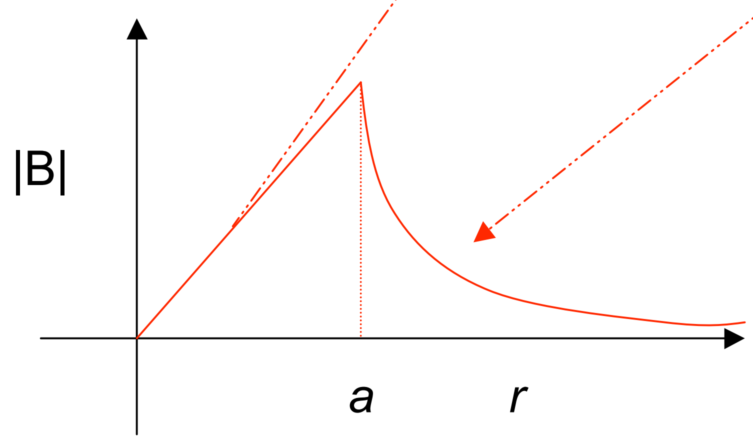
- Now make the wire ‘fat’ (of radius *a*), carrying a total current *I*

- Inside the wire

$$2\pi r|B| = \mu_0 \pi r^2 j \Rightarrow |B| = \frac{\mu_0 r j}{2}$$

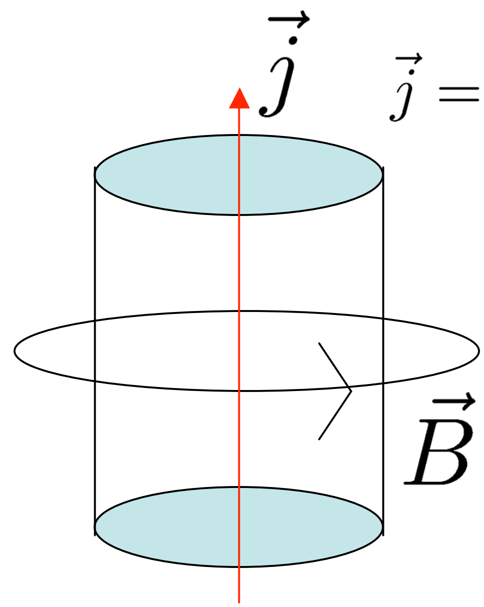
- Outside the wire

$$2\pi r|B| = \mu_0 \pi a^2 j \Rightarrow |B| = \frac{\mu_0 a^2 j}{2r}$$



- Inside, $\vec{B}$ has a similar field to $\vec{v}$ in solid-body rotation, where

$$\vec{\omega} \text{ maps over to } \frac{\mu_0 \vec{j}}{2} \Rightarrow \text{curl}\vec{B} = \mu_0 \vec{j} \text{ as expected}$$

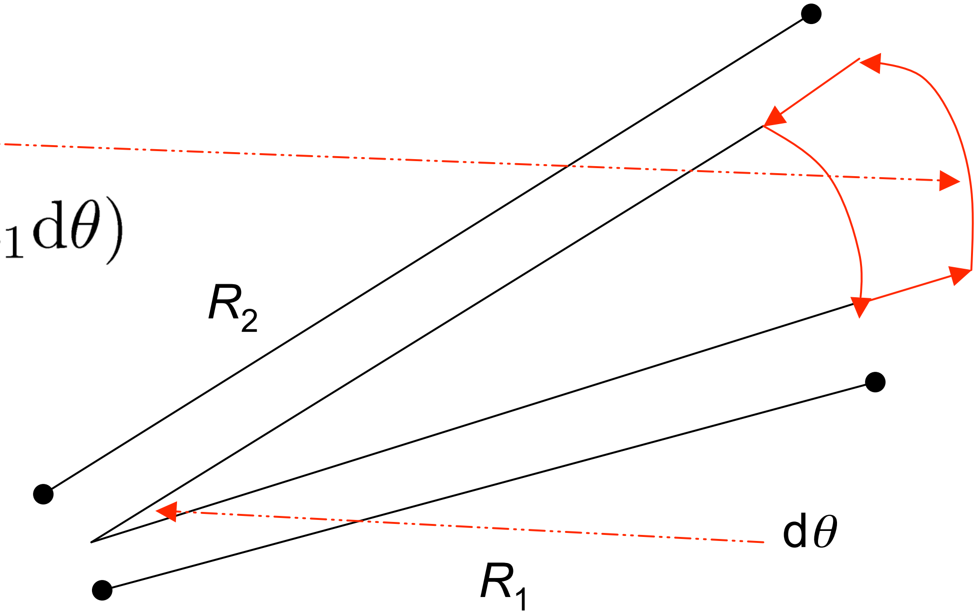


$$\vec{j} = (0, 0, \frac{I}{\pi a^2})$$

z axis along the wire, *r* is radial distance from the *z* axis

- Outside wire: *z* component from loops in the *xy* plane

$$\oint \vec{B}.\mathrm{d}\vec{l} = (B_2 R_2 \mathrm{d}\theta - B_1 R_1 \mathrm{d}\theta)$$



- *z* component is zero, because

$$B \propto \frac{1}{r}, Br = \mathrm{constant}$$

- and *x and y* components are also

$$\mathrm{since}\ \vec{B}.\mathrm{d}\vec{l} = 0$$

in the *yz* and *xz* planes

- Therefore $$\mathrm{curl}\vec{B} = \vec{0}\ \mathrm{outside\ the\ wire}$$

- Note that region where field is ***conservative*** ($\mathrm{curl}\vec{B} = 0$) is not ***simply connected:*** loop cannot be shrunk to zero without going inside wire (where field is not conservative).