Lecture 12: 2nd-order Vector Operators

1st \ 2nd	grad	div	curl
grad	$\times$	OK	$\times$
div	∇^2	$\times$	0
curl	$\vec{0}$	$\times$	OK

Lecture 11

$\times$ meaningless

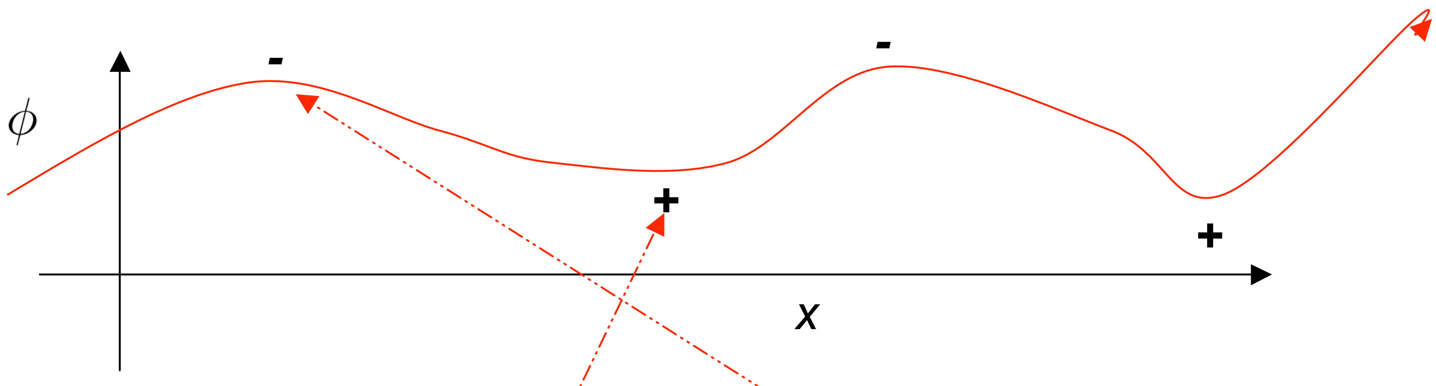
$$\nabla^2 = \vec{\nabla}.\vec{\nabla} \equiv \mathbf{DEL-SQUARED} \equiv \mathbf{LAPLACIAN}$$

Laplace's Equation $\nabla^2 \phi = 0$ is one of the most important in physics

- For a scalar function (potential) ϕ

$$\nabla^2 = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$$

- is the “second derivative” in 3D, a well-defined ***scalar function***
- Imagine first in 1D



$$\frac{\mathrm{d}^2\phi}{\mathrm{d}x^2} = +\text{ve, where curve is concave and } -\text{ve, where curve is convex}$$

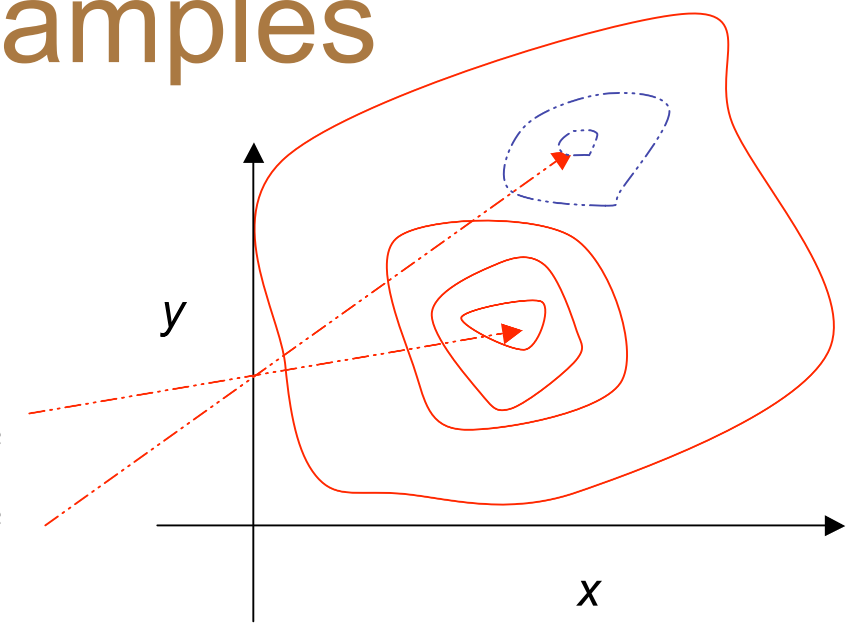
- If potential is lower than the average value in the surroundings then

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}x^2} = +\text{ve,} \;\; \text{and } -\text{ve, when } \phi(x) > \;\text{ average in the surroundings}$$

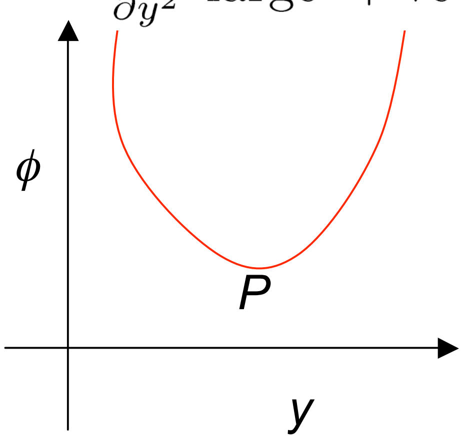
2D Examples

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2}$$

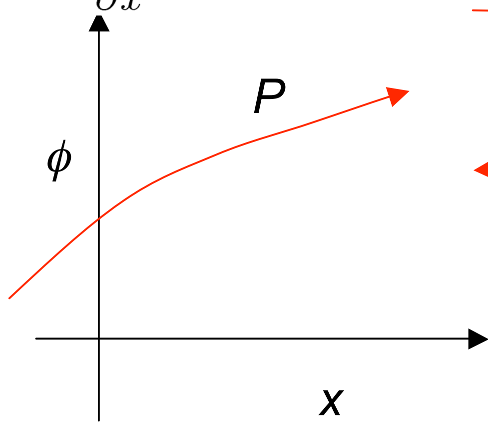
- ***Local maximum*** *(Q)* $\nabla^2 \phi = -\mathrm{ve}$
- ***Local minimum*** *(P)* $\nabla^2 \phi = +\mathrm{ve}$
- More complicated behaviour like a steep-sided valley

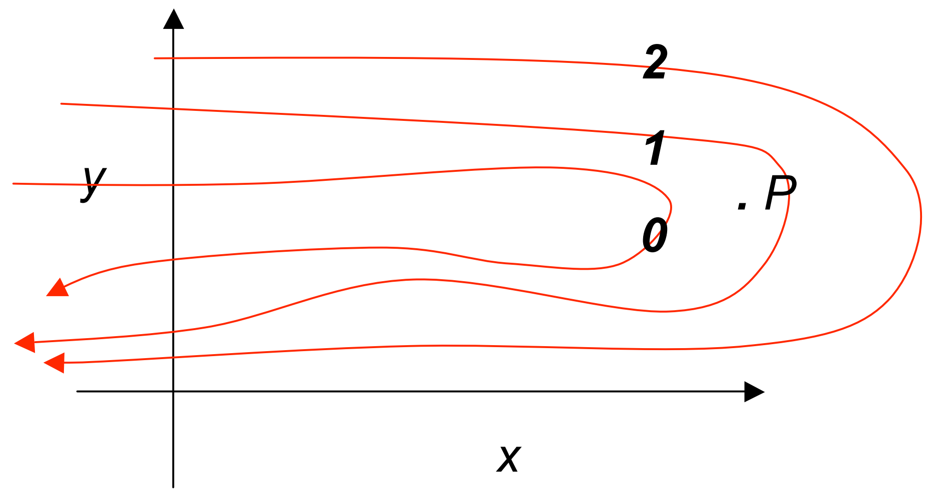


$\frac{\partial^2 \phi}{\partial y^2}$ large $+\mathrm{ve}$



$\frac{\partial^2 \phi}{\partial x^2}$ small $-\mathrm{ve}$





$\Rightarrow \nabla^2 \phi > 0$

ϕ is less than average value in the surroundings

3D Example

$$\mathrm{div}\vec{E} = \frac{\rho_f}{\epsilon_r \epsilon_0}$$

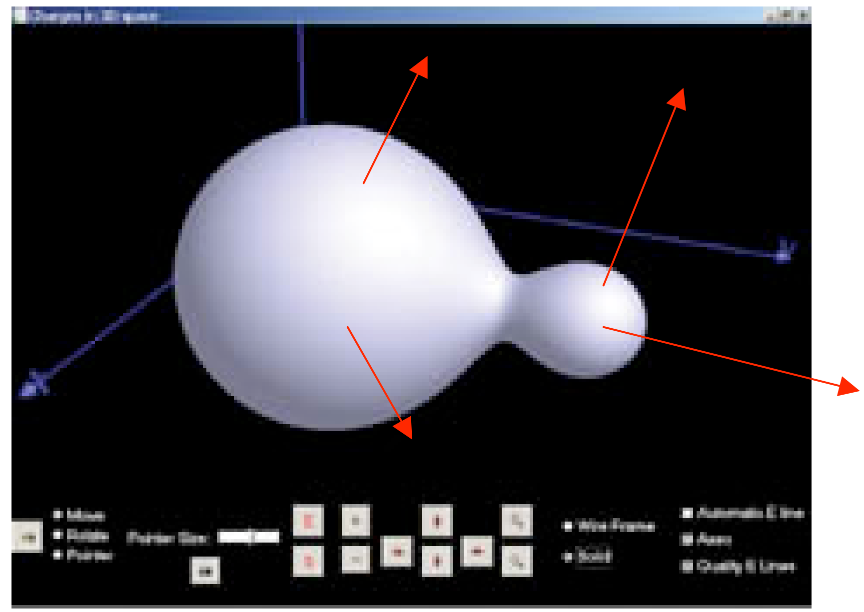
$$\vec{E} \text{ is conservative } \Rightarrow \vec{E} = -\vec{\nabla} V$$

$$\Rightarrow \nabla^2 V = \frac{-\rho_f}{\epsilon_r \epsilon_0}$$

Poisson's Equation

$$\text{in free space } \rho_f = 0 \Rightarrow \nabla^2 V = 0$$

Laplace's Equation



Equipotential around two unequal charges

- Note that Lapace's Equation implies there are no local maximum or minimum of the potential *V*

  ..... but there can be ***saddle points***

- Laplaces's Equation is very common in physics

Uniqueness Theorem

$$\text{Given } f \text{ and } (g \text{ or } h) \text{ then } V(x,y,z)$$

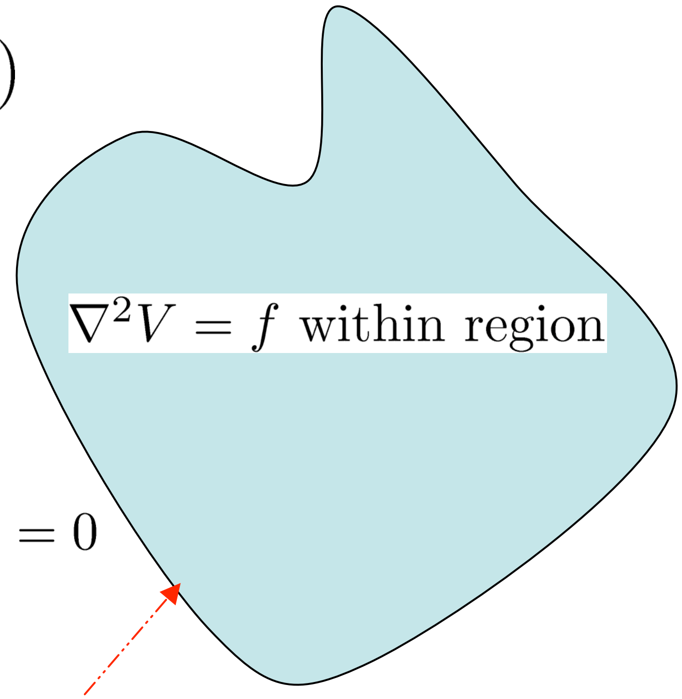
is ***UNIQUE*** to within a constant

- Consider two solutions V_1 and V_2 and define

$$\theta = V_1 - V_2 \text{ then } \nabla^2\theta = \nabla^2 V_1 - \nabla^2 V_2 = f - f = 0$$

- And, on surface $\theta = 0 \text{ or } \left(\frac{\partial\theta}{\partial n} = 0\right)$

- To progress further we need to insert θ into
  Green's First Theorem



$$V = g \quad (\text{or } \frac{\partial V}{\partial n} = h) \text{ on surface}$$

Boundary Condition:
of potential on surface (or on gradient of potential with respect to normal *n* to the surface)

Green's First Theorem

- Apply the ***Divergence Theorem*** to the vector $\phi\vec{\nabla}\psi$

$$\int \mathrm{d}V\,\vec{\nabla}.(\phi\vec{\nabla}\psi) = \oint \mathrm{d}\vec{S}.(\phi\vec{\nabla}\psi)$$

$$\Rightarrow \oint \mathrm{d}\vec{S}.(\phi\vec{\nabla}\psi) = \int \mathrm{d}V\left[\phi\nabla^2\psi + (\vec{\nabla}\phi).(\vec{\nabla}\psi)\right]$$

GREEN'S FIRST THEOREM

$$\text{put } \phi = \psi = V_1 - V_2 = \theta \Rightarrow \int_S \theta\frac{\partial\theta}{\partial n}\mathrm{d}S = \int_V \left[\theta\nabla^2\theta + (\vec{\nabla}\theta)^2\right]\mathrm{d}V$$

- One of these two is zero on surface, and this is zero

$$\Rightarrow \int_V (\vec{\nabla}\theta)^2\mathrm{d}V = 0 \Rightarrow \vec{\nabla}\theta = 0 \Rightarrow \theta = \text{ constant or } V_1 - V_2 \text{ is a constant}$$

- $V_1 = V_2$ throughout volume if $V_1 = V_2$ on the surface; the uniqueness theorem

Other Possible Combinations

- grad div

$$\vec{\nabla}(\vec{\nabla}.\vec{A}) = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\left[\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right]$$

$$= \left(\frac{\partial^2 A_x}{\partial x^2} + \frac{\partial^2 A_y}{\partial x \partial y} + \frac{\partial^2 A_z}{\partial x \partial z}, \frac{\partial^2 A_x}{\partial y \partial x} + \frac{\partial^2 A_y}{\partial y^2} + \frac{\partial^2 A_z}{\partial y \partial z}, \frac{\partial^2 A_x}{\partial z \partial x} + \frac{\partial^2 A_y}{\partial z \partial y} + \frac{\partial^2 A_z}{\partial z^2},\right)$$

$$\neq \vec{\nabla}^2(A_x, A_y, A_z) \equiv (\nabla^2 A_x, \nabla^2 A_y, \nabla^2 A_z)$$

- curl curl

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \vec{\nabla}(\vec{\nabla}.\vec{A}) - \vec{\nabla}^2 \vec{A}$$

which is not necessarily zero: we will prove this equation in Lecture 14