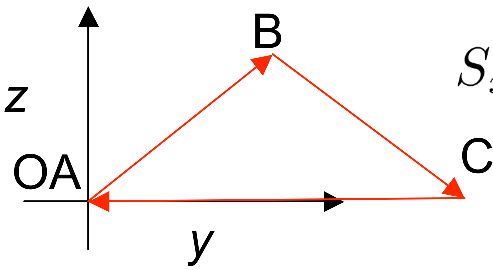
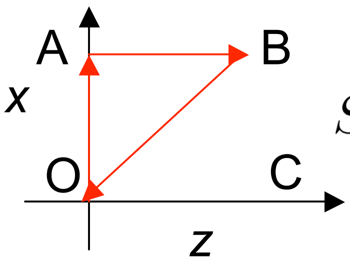
Lecture 13: Advanced Examples

- Selected examples taken from Problem Set 4
- ***Magnetic Potential*** (as an example of working with non-conservative fields)
- Plus ***Maxwell's Equations*** (as a concise example of the power of vector calculus)

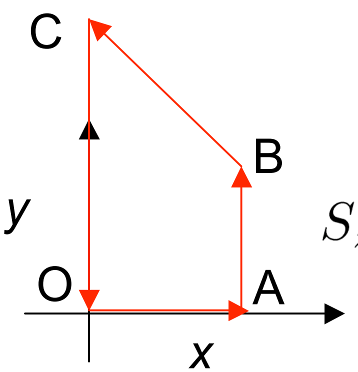
Vector Areas (Problem 4.1)



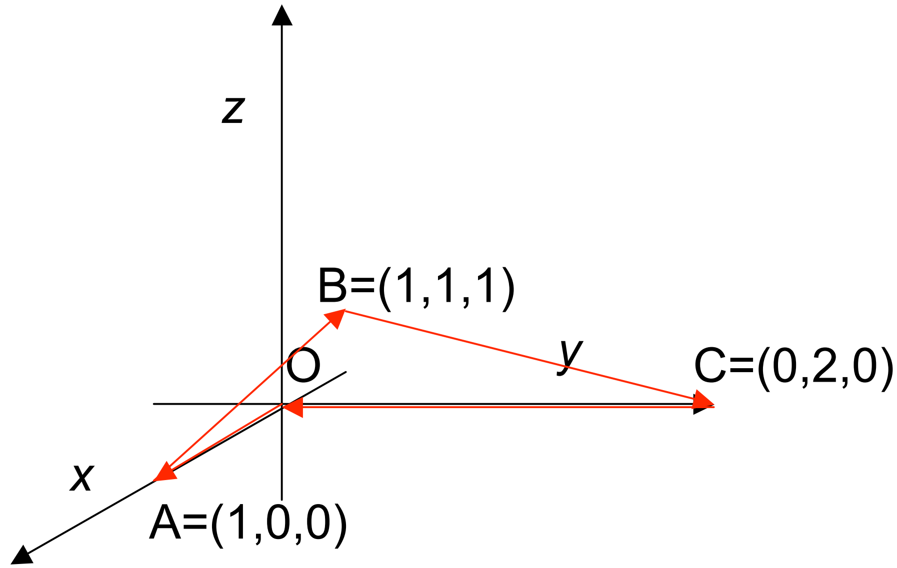
$$S_x = -1$$



$$S_y = -\frac{1}{2}$$



$$S_z = +\frac{3}{2}$$



Vector Area of Triangle OAB

$$\vec{S_1} = \frac{1}{2}(1,0,0) \times (1,1,1) = (0, -\frac{1}{2}, \frac{1}{2})$$

Vector Area of Triangle OBC

$$\vec{S_2} = \frac{1}{2}(1,1,1) \times (0,2,0) = (-1,0,1)$$

$$\vec{S} = (-1, -\frac{1}{2}, \frac{3}{2}) = \vec{S_1} + \vec{S_2}$$

$$|\vec{S}| = \sqrt{\frac{7}{2}}$$

Maximum Projected Area

$$\vec{S} \cdot \sqrt{\frac{1}{2}}(0 - 1, 1) = \sqrt{2}$$

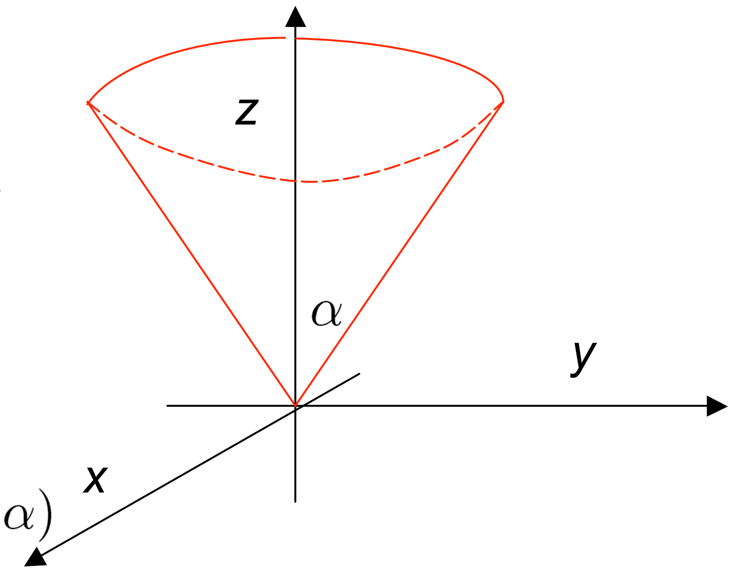
Area Projected in direction of this unit vector

Solid Angle (Problem 4.2)

- Angle (in ***radians***) is d*S/r* where *S* is a circle of radius *r* centred on the origin
- ***Solid Angle*** (in ***steradians***) is d*A*/*r*^{2} where d*A* is a sphere of radius *r* centred on the origin

$$\Omega = \int\int_A \frac{\vec{r}\cdot \mathrm{d}\vec{S}}{|\vec{r}|^3}$$

$$= \int_{\theta=0}^{\theta=\alpha}\int_{\phi=0}^{\phi=2\pi} \frac{r^2 \sin\theta}{r^2}\mathrm{d}\theta \mathrm{d}\phi = 2\pi(1-\cos\alpha)$$



- Applying divergence theorem to red volume

$$\Omega = \int\int\int_V \vec{\nabla}\cdot(f\vec{r})\,\mathrm{d}V \ \text{ where } f = \frac{1}{|\vec{r}|^3}$$

since on sides of closed volume, field is everywhere parallel to the sides so surface integrals are zero

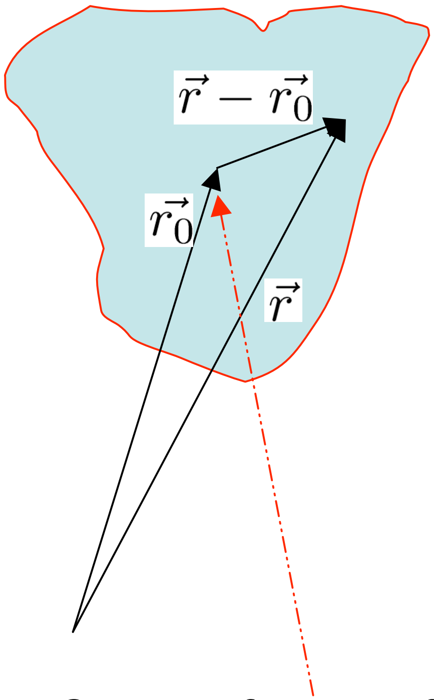
- $\text{and } \vec{\nabla}\cdot(f\vec{r}) = 3f + rf' = 0 \quad [\text{see Lecture10}]$

- Solution to this "paradox": divergence is non-zero at the origin: applying definition of divergence to small sphere around origin $\vec{\nabla}\cdot(f\vec{r})|_{\text{origin}}\mathrm{d}V = 4\pi$ and only a fraction $\frac{1-\cos\alpha}{2}$ of these field lines emerge into the cone; therefore divergence theorem works, if origin (source of field lines) is properly treated

Divergence (Problem 4.3)

- Heat up body so it expands by linear factor

$$\alpha = 1 + \delta\alpha \approx 1$$



$$\vec{r} \longrightarrow \vec{r'} = \vec{r} + \vec{h}(\vec{r})$$

$$\vec{r'} = \alpha(\vec{r} - \vec{r_0}) + \vec{r_0}$$
$$\Rightarrow \vec{h} = (\alpha - 1)(\vec{r} - \vec{r_0})$$
$$\Rightarrow \mathrm{div}(\vec{h}) = \vec{\nabla}.\vec{h} = (\alpha - 1)\vec{\nabla}.\vec{r} = 3(\alpha - 1)$$

- Note fractional increase in volume is

$$= \alpha^3 - 1 = (1 + \delta\alpha)^3 - 1 \approx 3\delta\alpha = 3(\alpha - 1)$$

- Divergence is a measure of local expansion/contraction

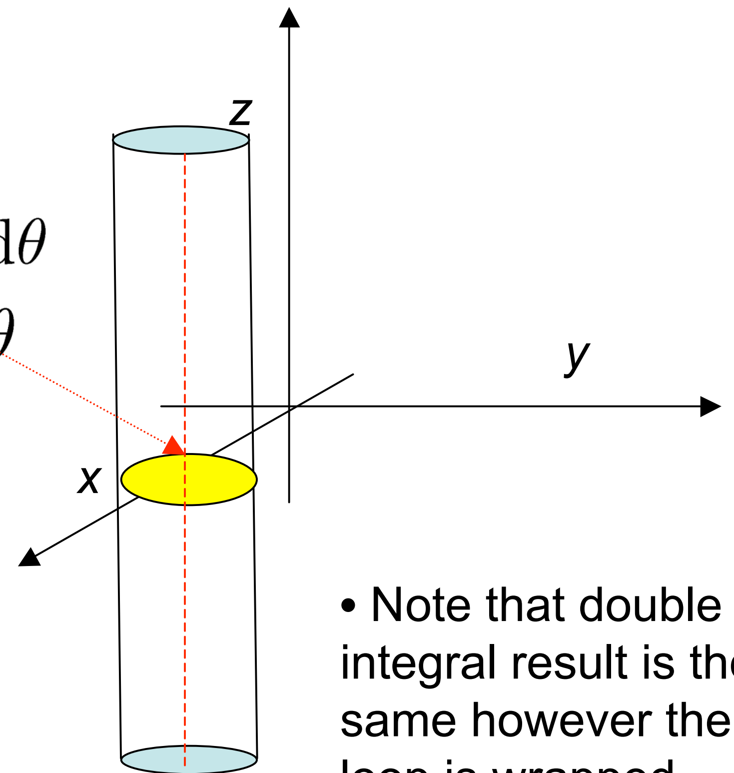
Curl (Problem 4.6)

$$\text{Vector Field } \vec{A} = (y, -x, 0) \text{ and Surface } (x-3)^2 + y^2 = 2$$

- Consider first, Stokes Theorem applied to a simple circular loop in the *z=0* plane

$$x = 3 + \sqrt{2}\cos\theta \Rightarrow \mathrm{d}x = -\sqrt{2}\sin\theta\mathrm{d}\theta$$

$$y = \sqrt{2}\sin\theta \Rightarrow \mathrm{d}y = \sqrt{2}\cos\theta\mathrm{d}\theta$$



Surface *S* has cylindrical sides + a circular cap (taken to be at +ve infinity)

$$(\vec{\nabla} \times \vec{A}) \cdot \mathrm{d}\vec{S} = 0$$

along sides of cylinder since $\vec{\nabla} \times \vec{A} = (0, 0, -2)$ is parallel to sides

$$\int\int(\vec{\nabla} \times \vec{A}) \cdot \mathrm{d}\vec{S} = -2\pi(\sqrt{2})^2 = -4\pi$$

on cap

$$\int\int_S \vec{\nabla} \times \vec{A} \cdot \mathrm{d}\vec{S} = \int \vec{A} \cdot \mathrm{d}\vec{r} =$$

$$\int_{\theta=0}^{\theta=2\pi} (\sqrt{2}\sin\theta, -3 - \sqrt{2}\cos\theta, 0) \cdot (-\sqrt{2}\sin\theta\mathrm{d}\theta, \sqrt{2}\cos\theta\mathrm{d}\theta, 0)$$

$$= -2\int_0^{2\pi} (\sin^2\theta + \cos^2\theta - 3\sqrt{2}\cos\theta)\mathrm{d}\theta = -4\pi$$

- Note that double integral result is the same however the loop is wrapped around the cylinder, so result is always

$$-4\pi$$

Magnetic Potential

- In the curl-free region can define a magnetic potential ϕ such that

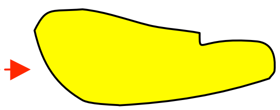
$$\vec{\nabla}\phi = \vec{B} \Rightarrow \phi = \frac{\mu_0 I}{2\pi}\theta,$$

where θ is the cylindrical polar angle.

Since In 2D polars $\vec{\nabla}\phi = \frac{\partial \phi}{\partial r}\vec{\hat{r}} + \frac{1}{r}\frac{\partial \phi}{\partial \theta}\vec{\hat{\theta}} \Rightarrow \vec{B} = \frac{\mu_0 I}{2\pi r}\vec{\hat{\theta}}$ as required by Ampere's Theorem

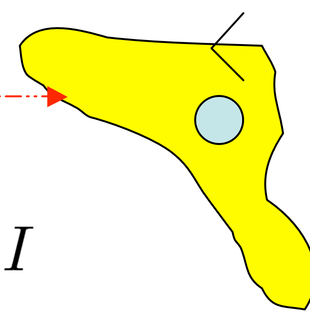
recalling that $\vec{\nabla}\phi \cdot \mathrm{d}\vec{r} = \vec{\nabla}\phi \cdot (\mathrm{d}r\vec{\hat{r}} + r\mathrm{d}\theta\vec{\hat{\theta}}) = \frac{\partial \phi}{\partial r}\mathrm{d}r + \frac{\partial \phi}{\partial \theta}\mathrm{d}\theta$

- This ***scalar potential*** must now be a ***multi-valued function*** of position



- (i) If loop doesn't enclose wire

$$\theta \rightarrow \theta \text{ and } \oint \vec{B}.\mathrm{d}\vec{l} = 0$$



wire carrying current out of the paper

- (ii) if loop goes round wire once

$$\theta \rightarrow \theta + 2\pi \Rightarrow \oint \vec{B}.\mathrm{d}\vec{l} = \phi_2 - \phi_1 = \mu_0 I$$

- (iii) if loop goes round wire *n* times

$$\theta \rightarrow \theta + n2\pi \Rightarrow \oint \vec{B}.\mathrm{d}\vec{l} = \mu_0 n I$$

Maxwell's Equations

$$\vec{\nabla}.\vec{B} = 0 \qquad (1)$$

"No magnetic monopoles"

$$\vec{\nabla}.\vec{D} = \rho_f, \text{ where } \vec{D} = \epsilon\epsilon_0\vec{E} \qquad (2)$$

"Gauss's Theorem"

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad (3)$$

"Faraday' s Law" since using Stoke's Theorem

$$\oint \vec{E}.\mathrm{d}\vec{l} = \int\int \vec{\nabla} \times \vec{E}.\mathrm{d}\vec{S} = -\frac{\partial}{\partial t}\int\int \vec{B}.\mathrm{d}\vec{S}$$

emf induced is rate of change of magnetic flux through the circuit

But ***"Ampere's Law"***

$$\vec{\nabla} \times \vec{H} = \vec{j}_f, \text{ where } \vec{B} = \mu\mu_0\vec{H} \text{ is incomplete}$$

because

$$\vec{\nabla}.(\vec{\nabla} \times \vec{H}) = \mathrm{div}\vec{j}_f = 0 \Rightarrow \frac{\partial \rho_f}{\partial t} = 0$$

if this were true capacitors couldn't charge up!

Recalling "continuity equation" $\vec{\nabla} \cdot \vec{j}_f = -\frac{\partial \rho_f}{\partial t}$ of Lecture 8

Maxwell's Displacement Current

- Maxwell added ***Displacement Current***

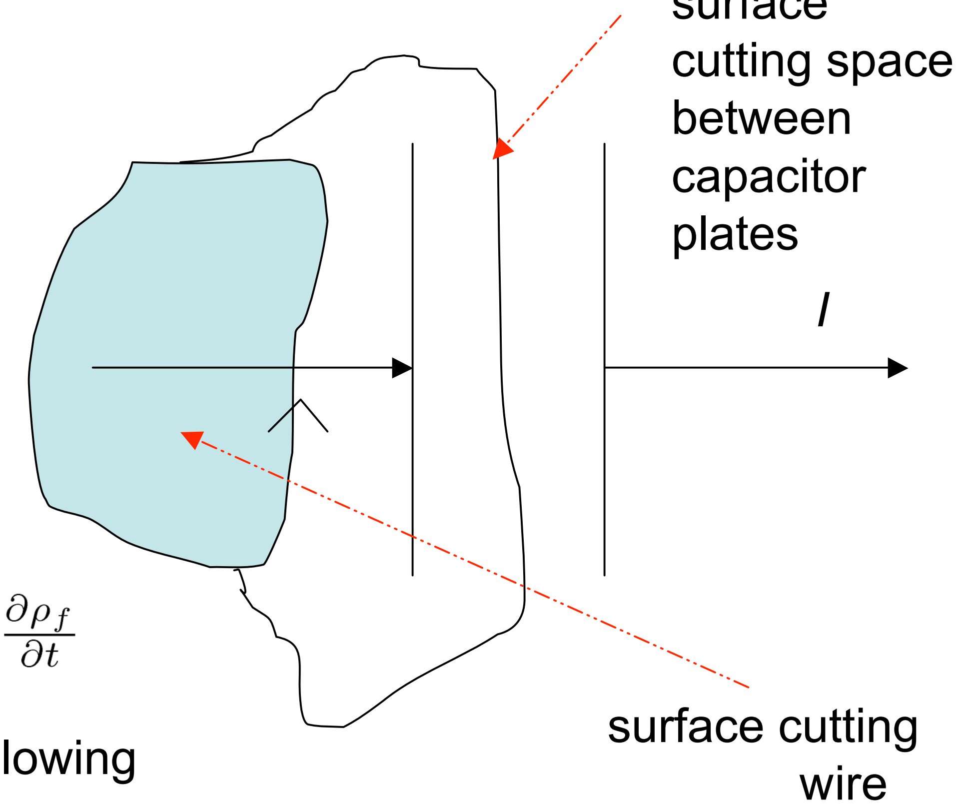
$$\vec{\nabla} \times \vec{H} = \vec{j}_f + \frac{\partial \vec{D}}{\partial t} \qquad (4)$$

- Now take $\vec{\nabla}.$

$$\Rightarrow \vec{\nabla}.\vec{j}_f + \vec{\nabla}.\frac{\partial \vec{D}}{\partial t} = 0$$

$$\Rightarrow \vec{\nabla}.\vec{j}_f = -\frac{\partial}{\partial t}(\vec{\nabla}.\vec{D}) = -\frac{\partial \rho_f}{\partial t}$$

- which is a continuity equation allowing charge to flow



... and, applying Stoke's Theorem to get $\oint \vec{B}.\mathrm{d}\vec{l}$, and hence $\vec{B}$, we know that we ***now*** get the same answer for $\vec{B}$ for a surface passing through the plates of the capacitor (with zero $\vec{j}_f$) as for a surface, with the same rim, cutting through the current-carrying wire (with non-zero $\vec{j}_f$)