Lecture 14: Einstein Summation Convention

- ***"In any expression containing subscripted variables appearing twice (and only twice) in any term, the subscripted variables are assumed to be summed over."***

- e.g. Scalar Product

$$\vec{a}.\vec{b} = \sum_{i=1}^{i=3} a_i b_i = a_1 b_1 + a_2 b_2 + a_3 b_3$$

- will now be written

$$\boxed{a_\theta b_\theta}$$

Good practice to use Greek letters for these ***dummy indices***

- In this lecture we will work in 3D so summation is assumed to be 1 - 3 but can be generalized to *N* dimensions

- Note dummy indices do not appear in the 'answer'. c.f.

$$I = \int f(\theta)\mathrm{d}\theta = \int f(x)\mathrm{dx} \text{ where } \theta, x \text{ are dummy variables}$$

Examples

- Total Differential

$$\mathrm{d}f = \frac{\partial f}{\partial x}\mathrm{d}x + \frac{\partial f}{\partial y}\mathrm{d}y + \frac{\partial f}{\partial z}\mathrm{d}z \equiv \vec{\nabla}.\mathrm{d}\vec{r}$$

becomes

$$\boxed{\mathrm{d}f = \frac{\partial f}{\partial x_\theta}\mathrm{d}x_\theta}$$

where $x_\alpha = (x_1, x_2, x_3) \equiv (x, y, z)$

and ***'free index'*** α runs from 1-3 here (or 1-*N* in and *N*-dimensional case)

- Matrix multiplication $\tilde{A} = \tilde{B}.\tilde{C}$

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}$$

$$\boxed{a_{ij} = \sum_k (b_{ik})(c_{kj}) \equiv b_{i\theta}c_{\theta j}}$$

Note the same 2 free indices on each side of the equation

- Trace of a matrix

$$\boxed{A_{11} + A_{22} + \ldots A_{nn} \equiv A_{\theta\theta}}$$

- Vector Product

need to define ***"alternating tensor"***

$$\epsilon_{ijk} \text{ a set of 27 numbers } \epsilon_{111}, \epsilon_{112} \ldots$$

$\epsilon_{ijk} = 0$ if any 2 indices are same — 21 cases of zero!

$\epsilon_{ijk} = +1$ if (i, j, k) are $(1, 2, 3)$ in cyclic order — 3 cases

$\epsilon_{ijk} = -1$ if (i, j, k) are $(2, 1, 3)$ in cyclic order — 3 cases

then (it's not that hard, honest!)

$$\boxed{[\vec{a} \times \vec{b}]_i \equiv \epsilon_{i\theta\phi} a_\theta b_\phi}$$

- There is a “simple” relation between the alternating tensor and the Kronecker delta

$$\epsilon_{\theta jk}\epsilon_{\theta lm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{lk}$$

- The proof is simply the evaluation of all 81 cases! (although symmetry arguments can make this easier).

- If you can get the hang of this, this provides the fastest and most reliable method of proving vector identities

- Once written down in this form, the order of the terms only matters if they include differential operators (which only act on things to the right-hand-side of them).

Example (ABACAB)

$$[\vec{a} \times (\vec{b} \times \vec{c})]_m = \epsilon_{m\alpha\beta} a_\alpha \underbrace{\epsilon_{\beta\gamma\delta} b_\gamma c_\delta}$$

rotating indices cyclically doesn't change sign

β-th component of $\vec{b} \times \vec{c}$

$$= \epsilon_{\beta m\alpha} \epsilon_{\beta\gamma\delta} a_\alpha b_\gamma c_\delta = (\delta_{m\gamma}\delta_{\alpha\delta} - \delta_{m\delta}\delta_{\alpha\gamma}) a_\alpha b_\gamma c_\delta$$

$$= (a_\alpha c_\alpha) b_m - (a_\alpha b_\alpha) c_m = \left[(\vec{a}.\vec{c})\vec{b} - (\vec{a}.\vec{b})\vec{c} \right]_m$$

- Which constitutes a compact proof of the (hopefully familiar) ***"ABACAB"*** formula
- Note we have made much use of the 'obvious' identity

$$\boxed{\delta_{i\theta} a_\theta = a_i}$$

Further Examples

- curl (grad) $\vec{\nabla} \times (\vec{\nabla}\phi) = 0$

$$\epsilon_{m\alpha\beta} \underbrace{\frac{\partial}{\partial x_\alpha}\frac{\partial \phi}{\partial x_\beta}} = 0$$

because this term is anti-symmetric (changes sign if α and β are swopped)

but this term symmetric (stays same if α and β are swopped)

so terms ‘cancel in pairs’

- div (curl) $\vec{\nabla}.(\vec{\nabla} \times \vec{A}) = 0$

$$\frac{\partial}{\partial x_\theta}\epsilon_{\theta\alpha\beta}\frac{\partial}{\partial x_\alpha}A_\beta = \epsilon_{\theta\alpha\beta}\frac{\partial^2 A_\beta}{\partial x_\theta \partial x_\alpha} = 0$$

as above

- Trace of the unit matrix

$$\delta_{\theta\theta} = \delta_{11} + \delta_{22} + \delta_{33} = 3$$

Final Example

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \vec{\nabla}(\vec{\nabla}.\vec{A}) - \vec{\nabla}^2 \vec{A}$$

$$\text{where } \vec{\nabla}^2(A_x, A_y, A_z) \equiv (\nabla^2 A_x, \nabla^2 A_y, \nabla^2 A_z)$$

$$\epsilon_{m\alpha\beta} \frac{\partial}{\partial x_\alpha} \epsilon_{\beta\theta\phi} \frac{\partial}{\partial x_\theta} A_\phi = \epsilon_{\beta m\alpha} \epsilon_{\beta\theta\phi} \frac{\partial}{\partial x_\alpha} \frac{\partial}{\partial x_\theta} A_\phi$$

$$= (\delta_{m\theta}\delta_{\alpha\phi} - \delta_{m\phi}\delta_{\alpha\theta}) \frac{\partial}{\partial x_\alpha} \frac{\partial}{\partial x_\theta} A_\phi$$

$$= \frac{\partial}{\partial x_\alpha} \frac{\partial}{\partial x_m} A_\alpha - \frac{\partial}{\partial x_\alpha} \frac{\partial}{\partial x_\alpha} A_m$$

$$= \left[\vec{\nabla}(\vec{\nabla}.\vec{A}) - \vec{\nabla}^2 \vec{A}\right]_m \qquad \textbf{\textit{QED}}$$

- It’s not as hard as it first looks!