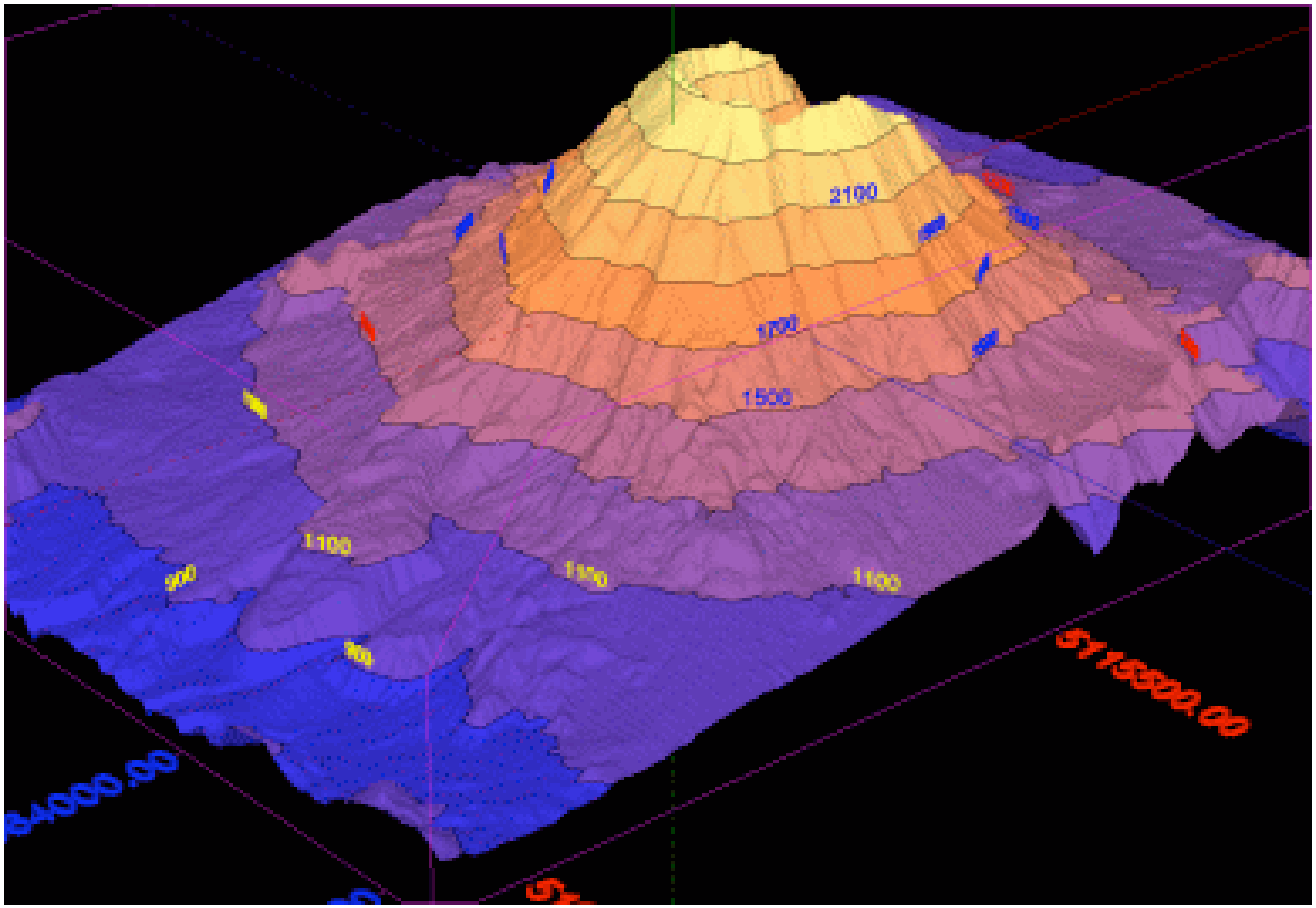
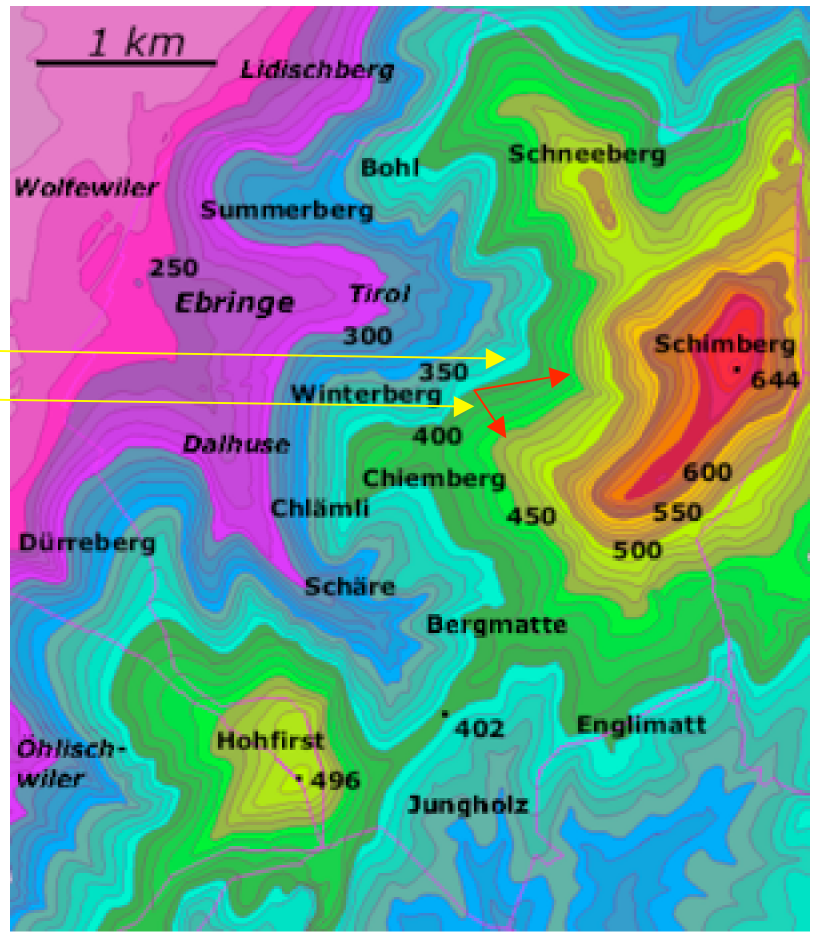
Lecture 2: Grad(ient)



2D Grad

- Need to extend idea of a gradient (df/dx) to 2D/3D functions
- Example: 2D scalar function h(x,y)
- Need "dh/dl" but dh depends on direction of dl (greatest up hill), define dl_{max} as short distance in this direction
- Define $\vec{\text{grad}}(h)$ magnitude $= |\frac{dh}{dl_{max}}|$
- Direction, that of steepest slope

$$
\begin{aligned}
dh &= \frac{\partial h}{\partial x}dx + \frac{\partial h}{\partial y}dy \\
&= \vec{\nabla}h.\vec{dl} \quad \text{if} \quad \vec{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)
\end{aligned}
$$



Vectors always perpendicular to contours

So if $\vec{\mathrm{d}l}$ is along a contour line

$$\mathrm{d}h = \vec{\nabla} h . \vec{\mathrm{d}l}_{\mathrm{cont}} = 0 \rightarrow \ \text{direction of } \vec{\nabla} h$$

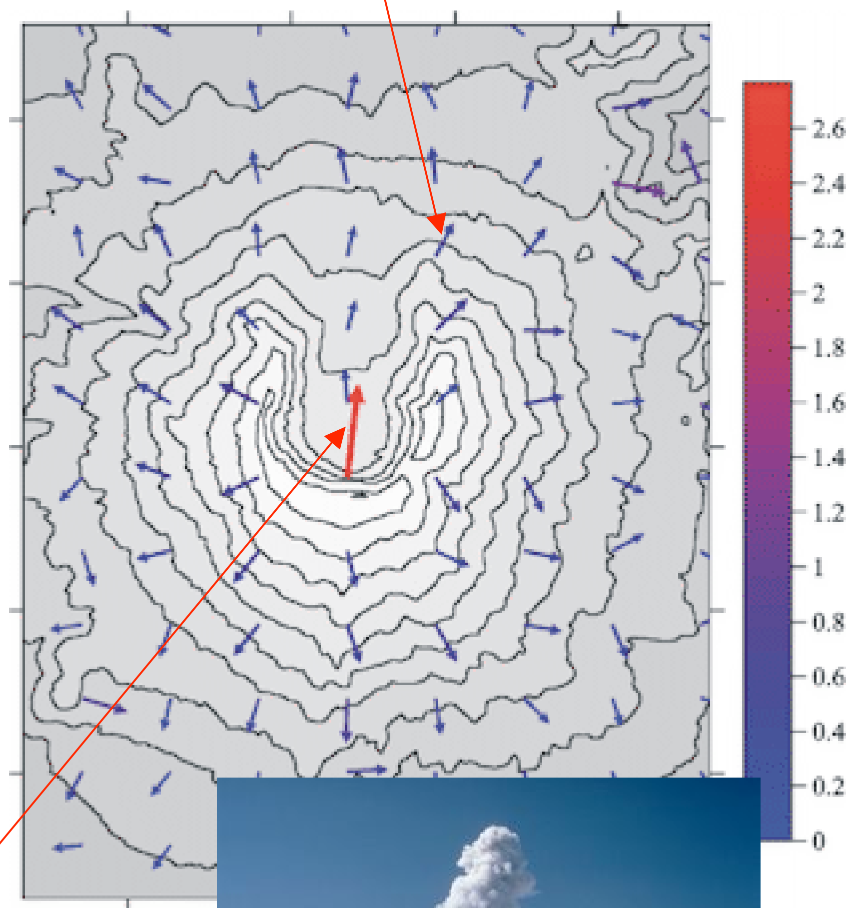
Is perpendicular to contours, ie up lines of steepest slope

And if $\vec{\mathrm{d}l}$ is along this direction

$$\mathrm{d}h = |\vec{\nabla} h| \mathrm{d}l_{\mathrm{max}} \rightarrow |\vec{\nabla} h| = \frac{\mathrm{d}h}{\mathrm{d}l_{\mathrm{max}}}$$

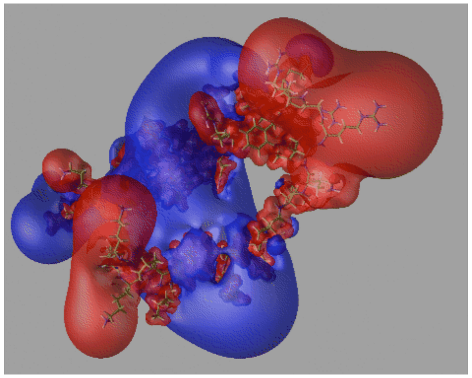
$$\vec{\mathrm{grad}}(h) = \vec{\nabla} h = \left(\frac{\partial \mathrm{h}}{\partial x}, \frac{\partial \mathrm{h}}{\partial y} \right)$$

The vector field shown is of $-\vec{\nabla} h$



Magnitudes of vectors greatest where slope is steepest

3D Grad



$$\mathrm{d}\phi = \frac{\partial \phi}{\partial x}\mathrm{d}x + \frac{\partial \phi}{\partial y}\mathrm{d}y + \frac{\partial \phi}{\partial z}\mathrm{d}z$$
$$= (\vec{\nabla}\phi).\mathrm{d}\vec{l}$$

And again $\vec{\nabla}\phi$ gives a vector field where the vectors are everywhere Perpendicular to ***contour surfaces***, and $|\vec{\nabla}\phi|$ encodes information on how Rapidly ϕ changes with position at any point

$$\vec{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$$

Is the ***"grad"*** or ***"del" operator***

- It acts on everything to the right of it (or until a closing bracket)
- It is a normal ***differential operator*** so, e.g.

$$\vec{\nabla}(\phi\psi) = \phi\vec{\nabla}\psi + \psi\vec{\nabla}\phi$$ Product Rule

$$\vec{\nabla}(\phi(u)) = \frac{\partial \phi}{\partial u}\vec{\nabla}u$$ Chain Rule

Scalar functions

Inverse square law forces

- Scalar Potential $V = \frac{k}{|\vec{r}|}$

equipotentials (surfaces of constant V) Are spheres centred on the origin

$$\vec{\nabla} V = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \frac{k}{|\vec{r}|}$$

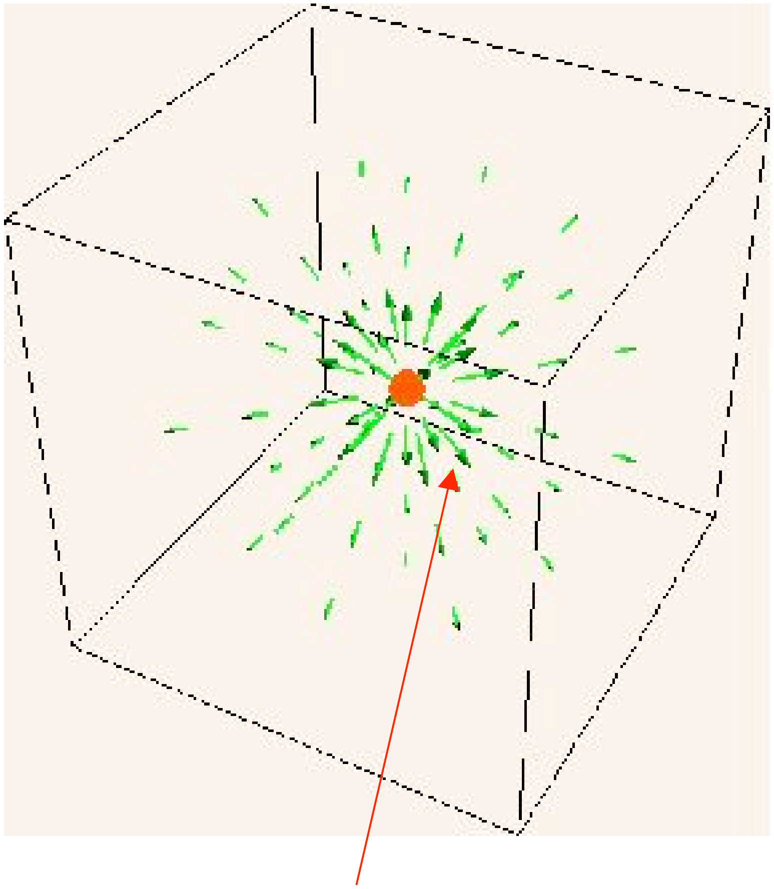
$$k \frac{\partial}{\partial x} \frac{1}{(x^2+y^2+z^2)^{1/2}} = -k \frac{1}{2} \frac{2x}{(x^2+y^2+z^2)^{3/2}}$$

And similarly for y, z components, so that

$$\vec{\nabla} V = -k \frac{\vec{r}}{|\vec{r}|^3} = -k \frac{\hat{\vec{r}}}{|\vec{r}|^2}$$

In ***electrostatics*** $k = Q/(4\pi\epsilon_0)$

In ***gravitation*** $k = -GM$



But note electric fields point away from +Q, needs sign convention: $\vec{E} = -\vec{\nabla} V$

Grad

$$\vec{\nabla}\phi = \vec{A}$$

"Vector operator acts on a scalar field to generate a vector field"

Tangent Planes

- Since $\vec{\nabla} f$ is perpendicular to contours, it locally defines direction of normal to surface

$$f(x, y, z) = A$$

- Defines a family of surfaces (for different values of A)
- $\vec{\nabla} f$ defines normals to these surfaces
- At a specific point $\vec{r_0} = (x_0, y_0, z_0)$

tangent plane has equation

$$\vec{r}.\vec{\nabla} f|_{(x_0,y_0,z_0)} = \vec{r_0}.\vec{\nabla} f|_{(x_0,y_0,z_0)}$$

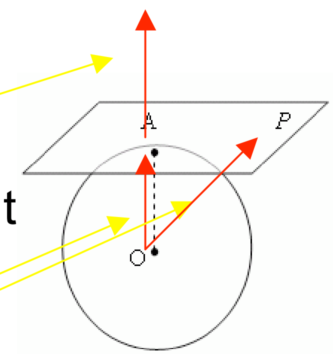


Fig. 95

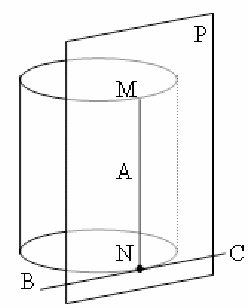


Fig. 96

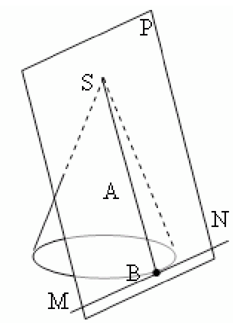
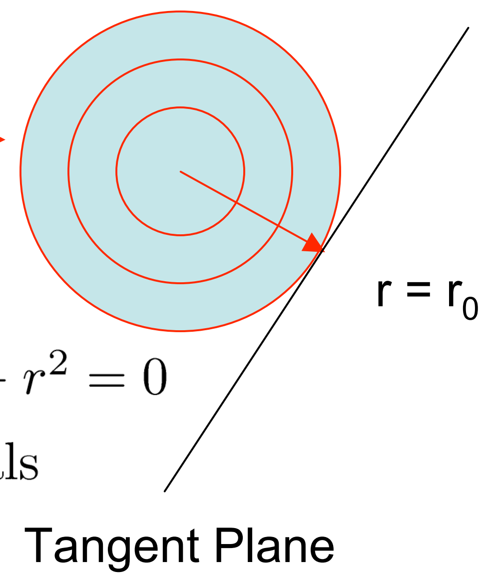


Fig. 97

Example of Tangent Planes

- Consider family of concentric spheres:

equipotentials of point charge



$$x^2 + y^2 + z^2 = r^2 \text{ or } f(x,y,z) = x^2 + y^2 + z^2 - r^2 = 0$$

$$\vec{\nabla} f = (2x, 2y, 2z) = 2\vec{r} \text{ gives direction of normals}$$

Not surprising as radial vectors are normal to spheres

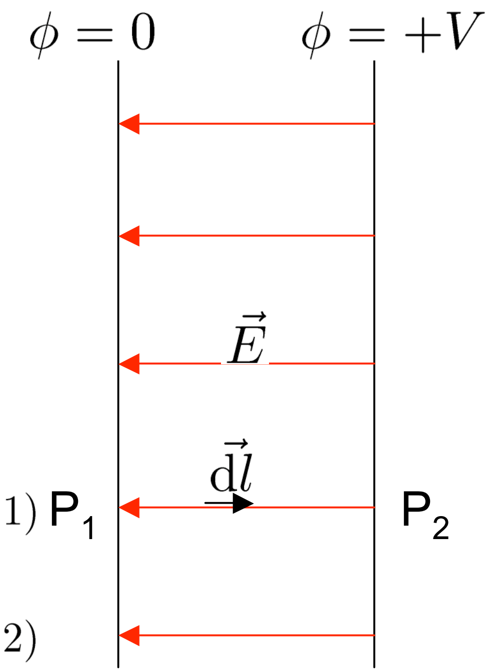
Tangent planes: $\vec{r}.\vec{r_0} = \vec{r_0}.\vec{r_0} \rightarrow (x,y,z).(x_0,y_0,z_0) = xx_0 + yy_0 + zz_0 = r_0^2$

$$\frac{\vec{r}.\vec{r_0}}{|\vec{r_0}|} = \vec{r}.\vec{\hat{r_0}} = r_0$$ → Perpendicular distance of origin to plane = r_0

Parallel Plate Capacitor

$$\vec{E} = -\vec{\nabla}\phi$$

Convention which ensures electric field points away from regions of positive scalar potential



Multiply by $\vec{\mathrm{d}l}$ and integrate from P_1 to P_2 in field

$$\int_{P_1}^{P_2} \vec{E}.\vec{\mathrm{d}l} = -\int \vec{\nabla}\phi.\vec{\mathrm{d}l} \qquad (1)$$

$$= -\int_{P_1}^{P_2} \left[\frac{\partial\phi}{\partial x}\mathrm{d}x + \frac{\partial\phi}{\partial y}\mathrm{d}y + \frac{\partial\phi}{\partial z}\mathrm{d}z\right] \qquad (2)$$

$$= -\int_{P_1}^{P_2} \mathrm{d}\phi = \phi(P_1) - \phi(P_2) \qquad (3)$$

Work done against force moving +Q from P_1 to P_2

$$-\int_{P_1}^{P_2} Q\vec{E}.\vec{\mathrm{d}l} = +VQ$$

Positive sign means charge has gained potential energy

This is a line integral

(see next lecture)