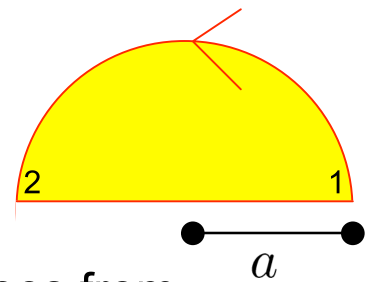
Lecture 3: Line Integrals

- We start with two (atypical) examples where ***integrand*** is
  (i) a scalar field, integrated w.r.t. a scalar



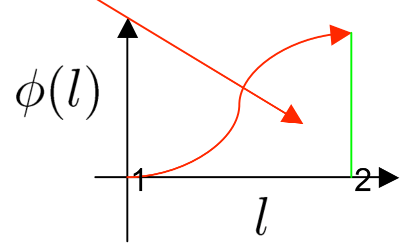
$$\int_L \phi \; \mathrm{d}l \text{ where } \phi(x,y) = (x-y)^2$$

and *L* goes from point 1 to point 2

$$\text{in polars } \phi(r,\theta) = a^2(1-\sin 2\theta) \qquad \mathrm{d}l = a \; \mathrm{d}\theta$$

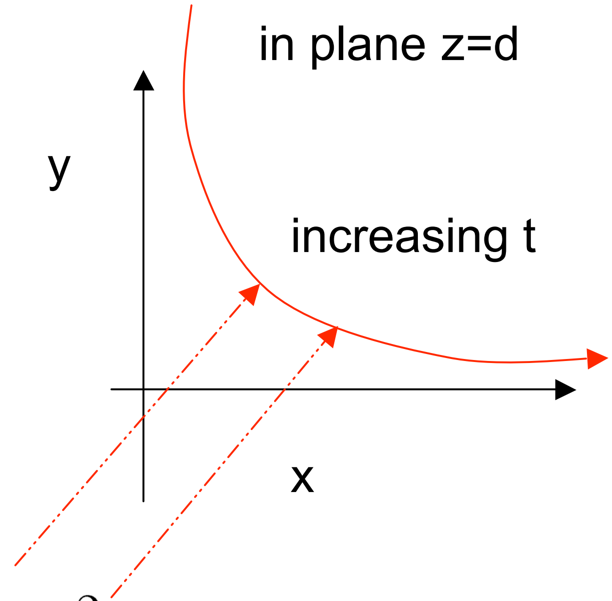
In general, it's the area under the following curve

$$\rightarrow \int_L \phi \; \mathrm{d}l = \int_0^{\pi} a^3(1-\sin 2\theta)\mathrm{d}\theta = \pi a^3$$



Physical example might be mass of a "chain" where ϕ is mass/unit length

- (ii) ***integrand*** is
  - (ii) a vector field, integrated w.r.t. a scalar



$$\int \vec{F}\ \mathrm{d}t$$

$$\vec{F} = (xy^2, 2, x)$$

$$\begin{aligned} x &= ct \\ y &= c/t \\ z &= d \quad \text{with } t = 1 \rightarrow 2 \end{aligned}$$

Integral is a vector

$$\begin{aligned} \int_1^2 \vec{F}\ \mathrm{d}t &= \left(\int_1^2 \frac{c^3}{t}\mathrm{d}t, \int_1^2 2\mathrm{d}t, \int_1^2 ct\mathrm{d}t \right) \\ &= (c^3 \ln 2, 2, 3c/2) \end{aligned}$$

Physical example might be “average” vector force on a moving particle

Common type of line integral

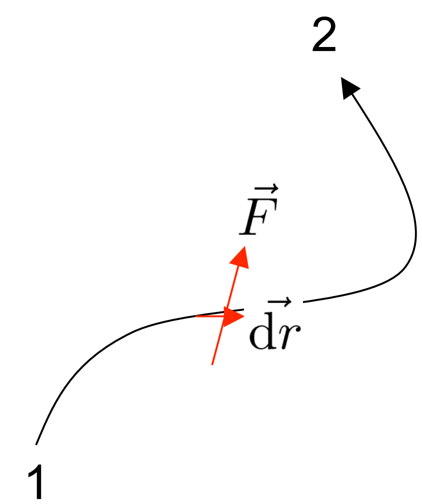
$$\int_1^2 \vec{A}.\mathrm{d}\vec{l} \text{ where } \vec{A} = [f(x,y), g(x,y)]$$

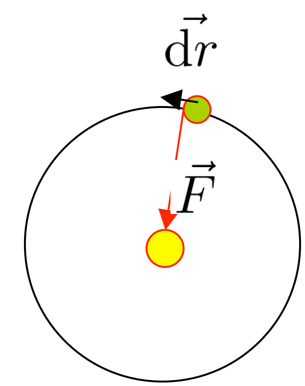
For example, work done by a force is $\vec{F} = \int_1^2 \vec{F}.\mathrm{d}\vec{r}$

Add up all the $\mathrm{d}w = \vec{F}.\mathrm{d}\vec{r}$ which is the work done by the force during a small displacement $\mathrm{d}\vec{r}$

Note that no work is done if the $\vec{F}$ and $\mathrm{d}\vec{r}$ are everywhere orthogonal

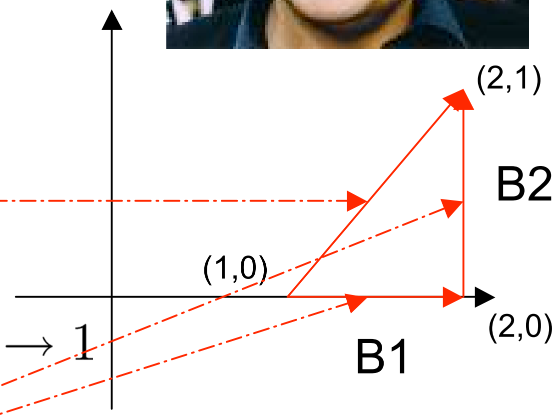
This underlies an interesting "problem" in quantum mechanics. Early theoretical ideas for quantum mechanics suggested that electrons moved in perfect circles around the Hydrogen nucleus. No work is done ($\vec{F}.\mathrm{d}\vec{r} = 0$) so how does the electron radiate energy? If $\vec{F}$ is not always perpendicular to $\mathrm{d}\vec{r}$, how does the electron avoid spiralling into the nucleus?





Example

$$\int_1^2 (y^3, x).(\mathrm{d}x, \mathrm{d}y)$$



Route A

$$y = \theta \quad x = \theta + 1$$

$$\mathrm{d}y = \mathrm{d}\theta \quad \mathrm{d}x = \mathrm{d}\theta \quad \theta : 0 \to 1$$

$$\int_{\theta=0}^1 \theta^3 \mathrm{d}\theta + \int_0^1 (\theta + 1)\mathrm{d}\theta = \left[\frac{\theta^4}{4} + \frac{\theta^2}{2} + \theta\right]_0^1 = \frac{7}{4}$$

Route B = B1+B2.

B1

$$x = \theta \quad y = 0$$

$$\mathrm{d}x = \mathrm{d}\theta \quad \mathrm{d}y = 0 \quad \theta : 1 \to 2$$

$$\int_1^2 (0)\mathrm{d}\theta + \int_1^2 \theta(0) = 0$$

B2

$$y = \theta \quad x = 2$$

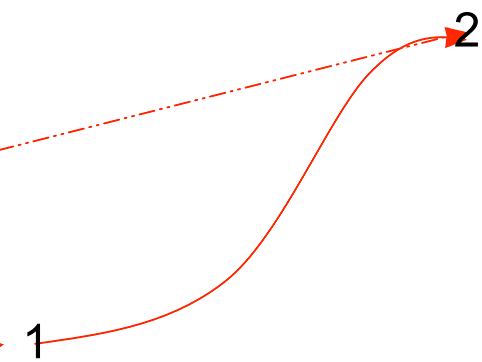
$$\mathrm{d}\theta = \mathrm{d}y \quad \mathrm{d}x = 0 \quad \theta : 0 \to 1$$

$$\int_0^1 \theta^3(0) + \int_0^1 2\mathrm{d}\theta = 2$$

Since $\frac{7}{4} \neq 2$ this ***line integral*** depends on the ***path of integration.***

Conservative fields

- Important difference between ***line integrals*** that (can) depend on both ***start/end points*** and ***path,*** and those that depend only on ***start/end points***
- Consider ***exact differential*** of some function ϕ

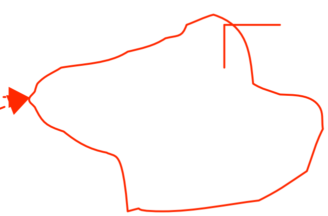


$$\mathrm{d}\phi = \frac{\partial \phi}{\partial x}\mathrm{d}x + \frac{\partial \phi}{\partial y}\mathrm{d}y = \vec{\nabla}\phi.\mathrm{d}\vec{l}$$

then $\displaystyle\int_{\text{point 1}}^{\text{point 2}} \mathrm{d}\phi = \phi_2 - \phi_1$ is ***independent of path***, and we can construct a contour map of $\phi(x,y)$

$$\int_1^2 \vec{A}.\mathrm{d}\vec{l} \text{ is independent of path } \equiv \text{vector field } \vec{A} = \vec{\nabla}\phi \text{ for some scalar field } \phi$$

- and vector field $\vec{A}$ is said to be ***Conservative***
- Around any closed loop, we begin and end at same point $\displaystyle\int_1^1 \vec{A}.\mathrm{d}\vec{l} = 0$



Examples

$$F(x,y) = xy^3 \rightarrow \mathrm{d}F = y^3\mathrm{d}x + 3y^2x\mathrm{d}y$$

Could write as $\vec{A}.\vec{\mathrm{dl}}$ where $\vec{A} = (y^3, 3y^2x)$

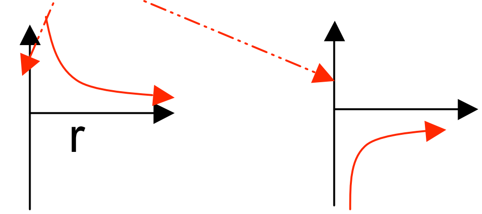
So $\int_1^2 (y^3\mathrm{d}x + 3y^2x\mathrm{d}y) = [xy^3]_1^2$ is independent of path

- Commonly in physics we have scalar ***potential*** functions (normally, but not exclusively) potentials are ***single-valued*** functions of position

Electric Field $\vec{E} = -\vec{\nabla}\phi \rightarrow \phi = -\int \vec{E}.\vec{\mathrm{dl}}$ and $\oint \vec{E}.\vec{\mathrm{dl}} = 0$ ϕ +ve near +ve Q

Gravitational Field $\vec{g} = -\vec{\nabla}\phi \rightarrow \phi = -\int \vec{g}.\vec{\mathrm{dl}}$ and $\oint \vec{g}.\vec{\mathrm{dl}} = 0$ ϕ -ve near +ve M

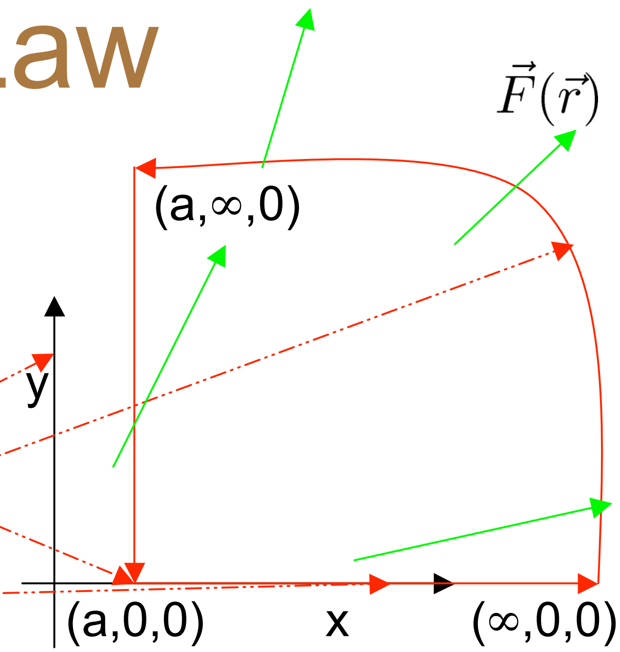
- In both cases $\phi|_\infty = 0$ defines ***zero point*** of potential



Inverse Square Law

$$\vec{F} = \frac{\hat{\vec{r}}}{|\vec{r}|^2}$$

Due to +ve charge at origin. Take another +ve charge from point 1 to infinity and back again (all in z=0 plane).



$$\text{Work done against force} = -\int \vec{F}.\mathrm{d}\vec{l}$$

Path A, along x axis

$x = \theta, \mathrm{d}x = \mathrm{d}\theta,\ \ \theta : a \rightarrow \infty$ $\quad \vec{F} = (\frac{1}{\theta^2}, 0, 0)\ \ \mathrm{d}\vec{l} = (\mathrm{d}x, 0, 0)$

So work done against force is $-\int_a^\infty \frac{1}{\theta^2}\mathrm{d}\theta = -\frac{1}{a}$

Path B, around large loop, force and path perpendicular $\vec{F}.\mathrm{d}\vec{l} = 0$ so no work done

Path C, parallel to y axis $y = \theta, \mathrm{d}y = \mathrm{d}\theta,\ \ \theta : \infty \rightarrow 0\ \ \vec{F} = (\frac{x}{r^3}, \frac{y}{r^3}, 0)\ \ \mathrm{d}\vec{l} = (0, \mathrm{d}y, 0)$

So work done against force is $-\int_\infty^0 \frac{\theta}{(\theta^2 + a^2)^{3/2}}\mathrm{d}\theta = \left[\frac{1}{(\theta^2 + a^2)^{1/2}}\right]_\infty^0 = +\frac{1}{a}$

Total work done is zero, as expected for a ***closed-loop line integral*** in a ***conservative*** vector field

Summary

In a Conservative Vector Field $\vec{A}$

$\int_1^2 \vec{A}.\mathrm{d}\vec{l}$ is independent of path $\equiv$ vector field $\vec{A} = \vec{\nabla}\phi$ for scalar potential ϕ

$$\mathrm{d}\phi = \frac{\partial\phi}{\partial x}\mathrm{d}x + \frac{\partial\phi}{\partial y}\mathrm{d}y + \frac{\partial\phi}{\partial z}\mathrm{d}z = \vec{\nabla}\phi.\mathrm{d}\vec{l}$$

$$\int_1^2 \vec{A}.\mathrm{d}\vec{l} = \mathrm{d}\phi = \phi_2 - \phi_1$$

Which gives an easy way of evaluating line integrals: regardless of path, it is difference of potentials at points 1 and 2.

$$\oint \vec{A}.\mathrm{d}\vec{l} = 0$$

Obvious provided potential is single-valued at the start and end point of the closed loop.