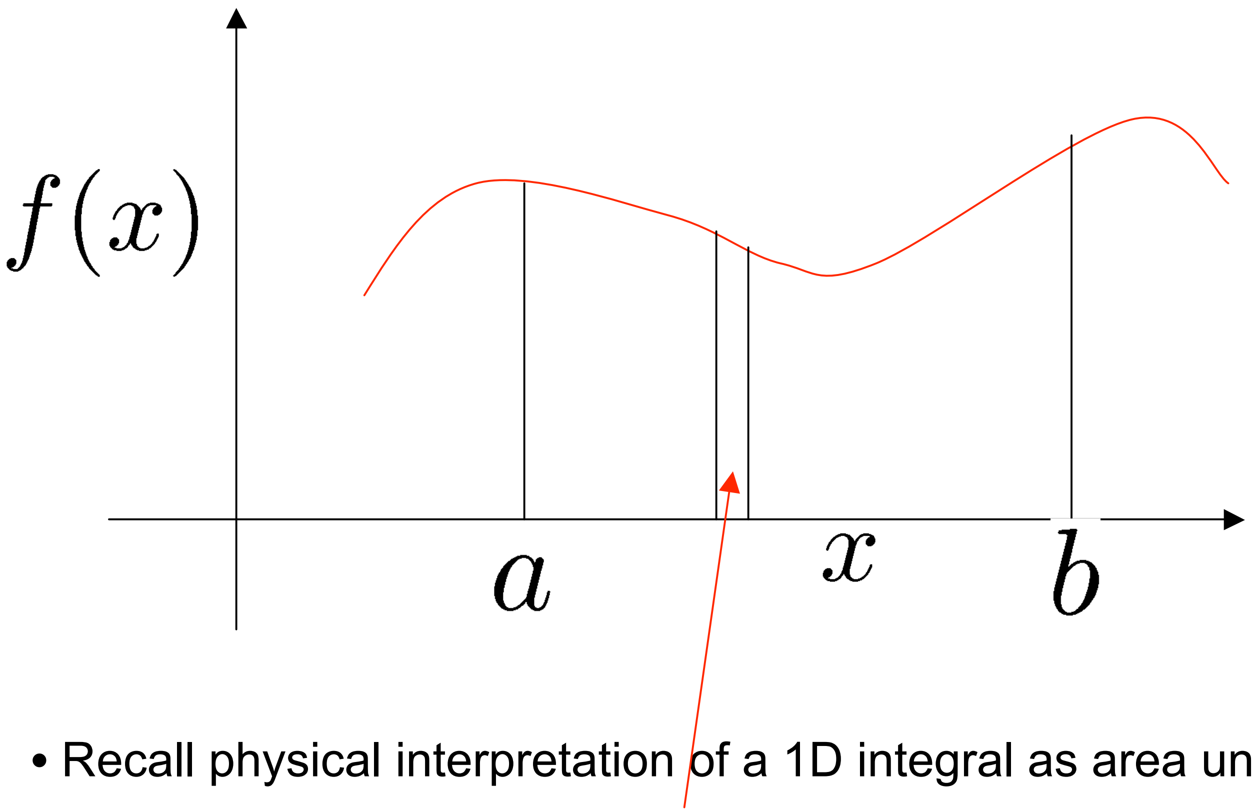
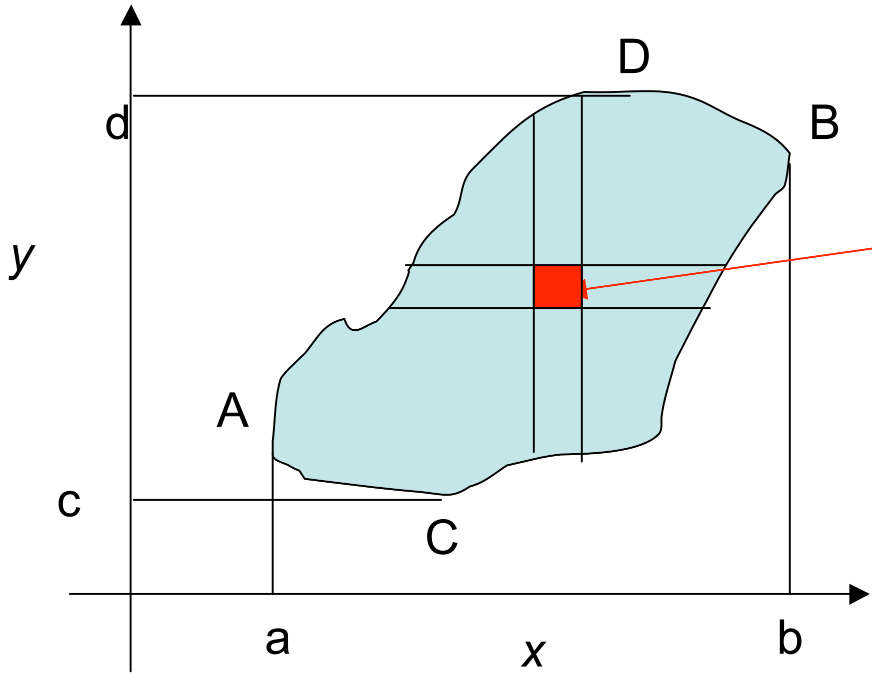
Lecture 4: Multiple Integrals

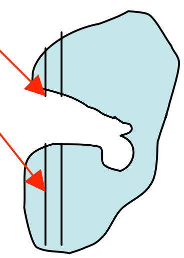


- Recall physical interpretation of a 1D integral as area under curve
- Divide domain (*a,b*) into *n* strips, each of width δr_k, *k*=1,2...,*n*
- As $\delta r \to 0$ and $n \to \infty$ we get $\int_a^b f(x)\mathrm{d}x = \sum_{k=1}^{k=n} f_{\delta r}\delta r$
- With value of function $f_{\delta r}$ constant within each strip

2D Integrals



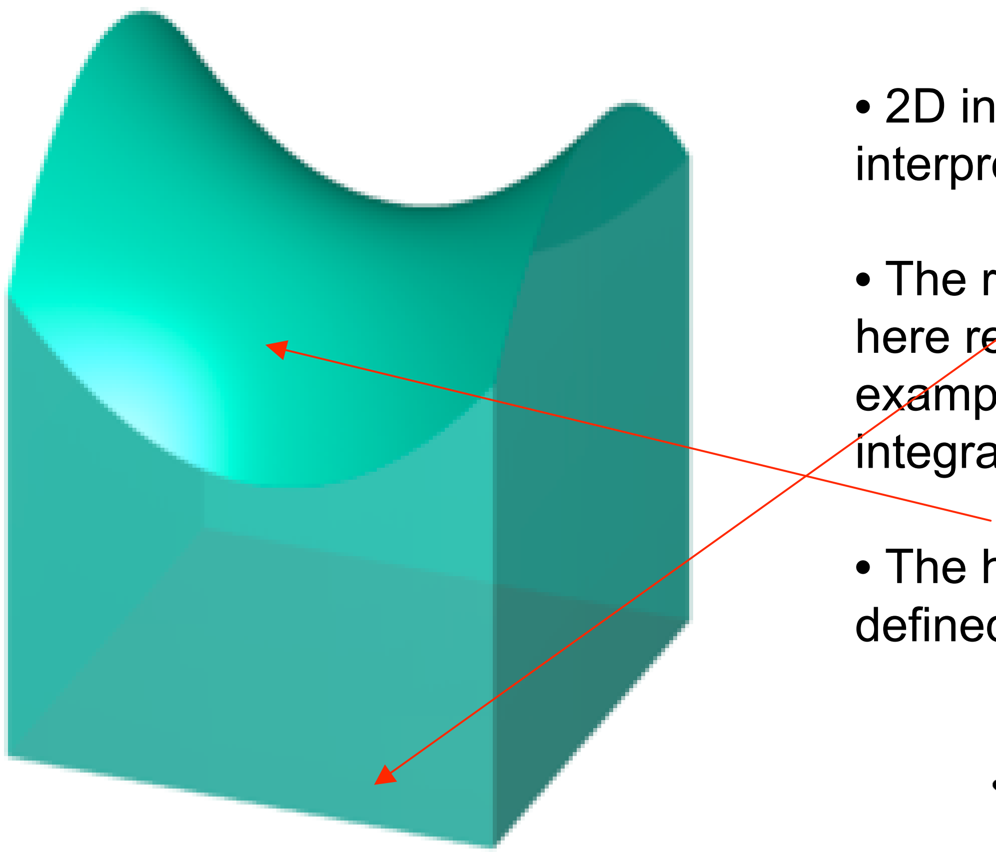
- Divide into *n* sub-regions, each of area δA_k, k=1,2...,n
- For small δA_k, value of function $f_{\delta A_k}$ is constant across the patch
- $y = g_1(x)$ is curve ACB
  $y = g_2(x)$ is curve ADB
  $x = h_1(y)$ is curve CAD
  $x = h_2(y)$ is curve CBD
- Complicated regions may need carving up



- Definition $\int\int_R f(x,y)\mathrm{d}A = \lim_{n\to\infty}\sum_{k=1}^{k=n} f_{\delta A_k}\delta A_k$
- Cartesian Grid $= \int_{x=a}^{x=b}\left[\int_{y=g_1(x)}^{y=g_2(x)} f(x,y)\mathrm{d}y\right]\mathrm{d}x$

or alternatively (equivalently) $= \int_{y=c}^{y=d}\left[\int_{x=h_1(x)}^{x=h_2(x)} f(x,y)\mathrm{d}x\right]\mathrm{d}y$

Physical Interpretation

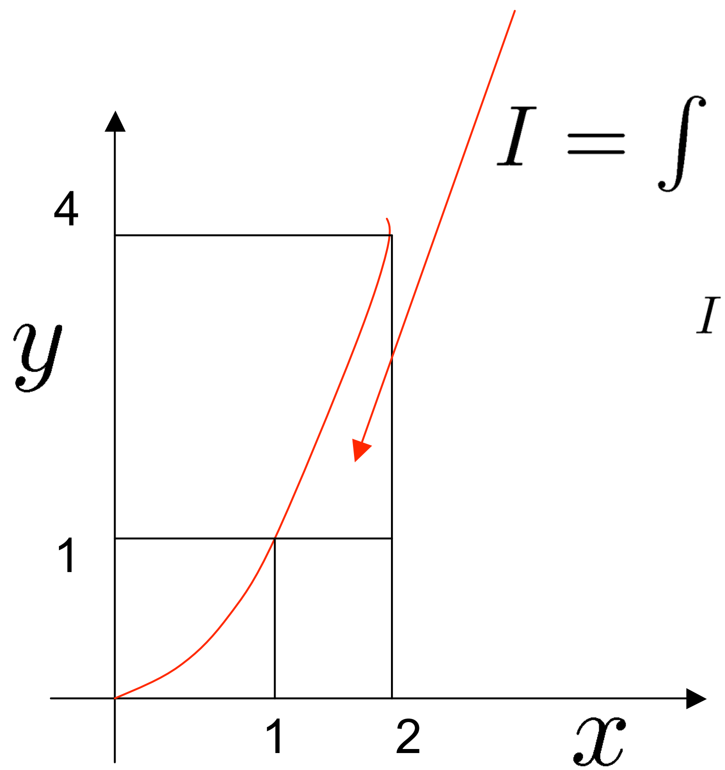


- 2D integrals can be interpreted as a volume
- The rectangular region here represents an example domain of integration
- The height of surface is defined by the value of

$$f(x, y)$$

Example

- R is a region bounded by $y=x^2$, $x=2$, $y=1$
- Calculate



$$I = \int\int_R (x^2 + y^2)\mathrm{d}x\mathrm{d}y$$

$$I = \int_1^2 \left[\int_1^{x^2} (x^2 + y^2)\mathrm{d}y\right] \mathrm{d}x$$

$$= \int_1^2 \left[x^2 y + \frac{y^3}{3}\right]_{y=1}^{y=x^2} \mathrm{d}x$$

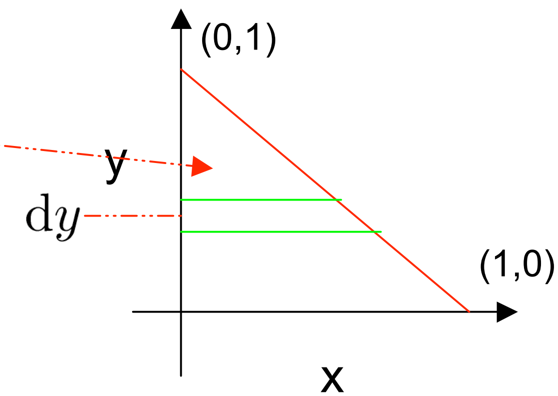
$$= \frac{1006}{105}$$

$$I = \int_1^4 \left[\int_{x=\sqrt{y}}^{x=2} (x^2 + y^2)\mathrm{d}x\right] \mathrm{d}y \qquad = \frac{1006}{105}$$

- Order of integration unimportant for well-behaved functions

Another Example

$$\int\int_S x^2 y \, \mathrm{d}x \mathrm{d}y \text{ over}$$



Do x first

$$\mathrm{d}y \left(\int_0^{1-y} x^2 y \, \mathrm{d}x \right) = \frac{y\mathrm{d}y}{3} \left[x^3 \right]_0^{1-y} = \frac{y\mathrm{d}y}{3} (1-y)^3$$

Then do y

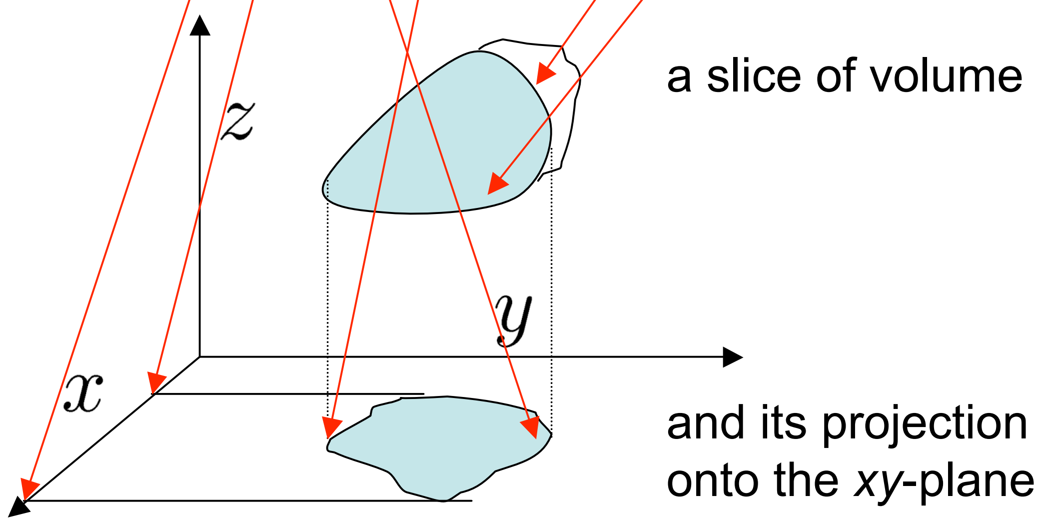
$$\int_0^1 \left(\frac{y}{3} - y^2 + y^3 - \frac{y^4}{3} \right) \mathrm{d}y = \frac{1}{60}$$

Exercise for student: obtain same answer by integrating in other order: e.g., y first.

3D Integrals

- Divide volume *V* into sub-volumes each of volume $\delta V_k, k = 1, 2...., n$
- Definition as per 2D but generalised to add up *n* volumes (rather than *n* areas)
- Using Cartesian grid

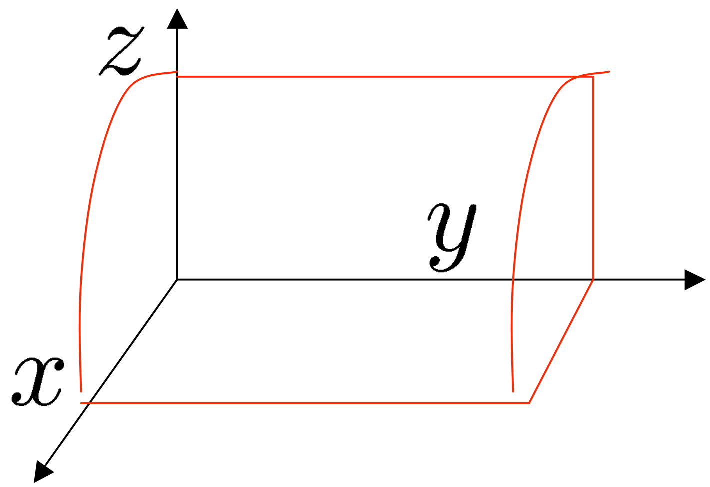
$$\int\int\int f(x,y,z)\mathrm{d}V = \int_{x=a}^{x=b}\left[\int_{y=g_1(x)}^{y=g_2(x)}\left(\int_{z=f_1(x,y)}^{z=f_2(x,y)} f(x,y,z)\mathrm{d}z\right)\mathrm{d}y\right]\mathrm{d}x$$



- Note care is needed if *V* has dimples

Simple Example (Cartesian)

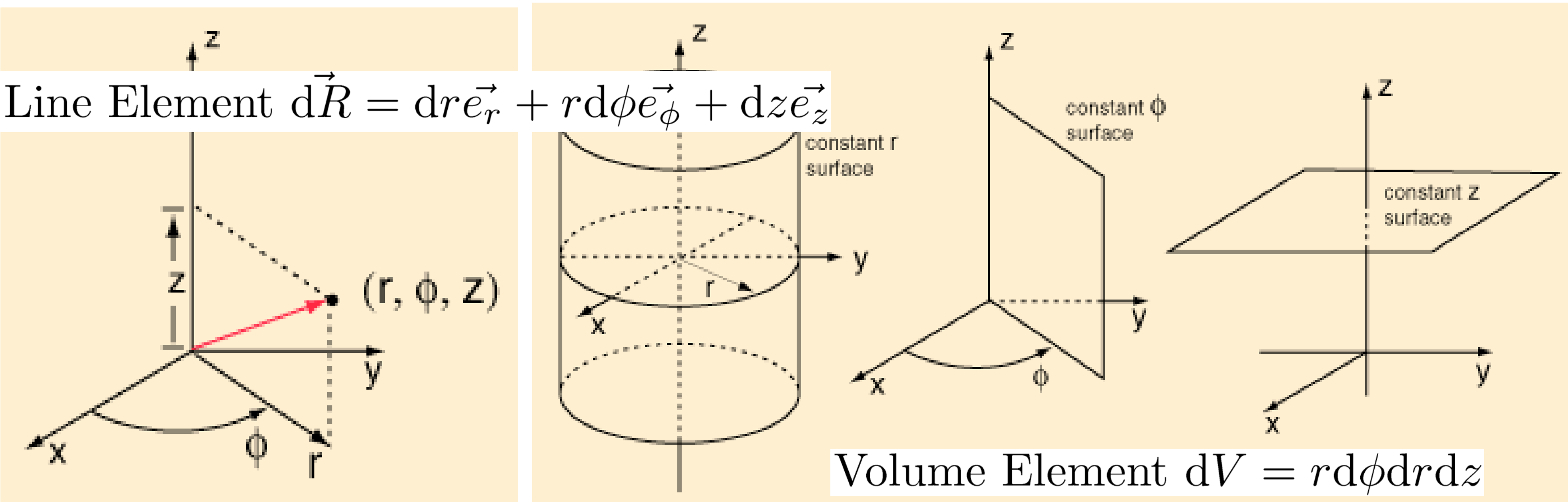
- Find the volume V bounded by the parabolic cylinder $z = 4 - x^2$ and the planes $x = 0, y = 0, y = 6, z = 0$



$$V = \int\int\int \mathrm{d}x\mathrm{d}y\mathrm{d}z = \int_{x=0}^{x=2}\int_{y=0}^{y=6}\int_{z=0}^{z=4-x^2} \mathrm{d}x\mathrm{d}y\mathrm{d}z$$

$$= 6[4x - \tfrac{1}{3}x^3]_0^2 = 32$$

Cylindrical Polar Coordinates



$$\text{Line Element } \mathrm{d}\vec{R} = \mathrm{d}r\vec{e_r} + r\mathrm{d}\phi\vec{e_\phi} + \mathrm{d}z\vec{e_z}$$

$$\text{Volume Element } \mathrm{d}V = r\mathrm{d}\phi\mathrm{d}r\mathrm{d}z$$

Note *r* is perpendicular distance from the cylinder axis; ϕ is the ***azimuthal angle***

$$x = r\cos\phi \qquad r = (x^2 + y^2)^{1/2}$$

$$y = r\sin\phi \qquad \phi = \arctan\frac{y}{x}$$

$$z = z \qquad z = z$$

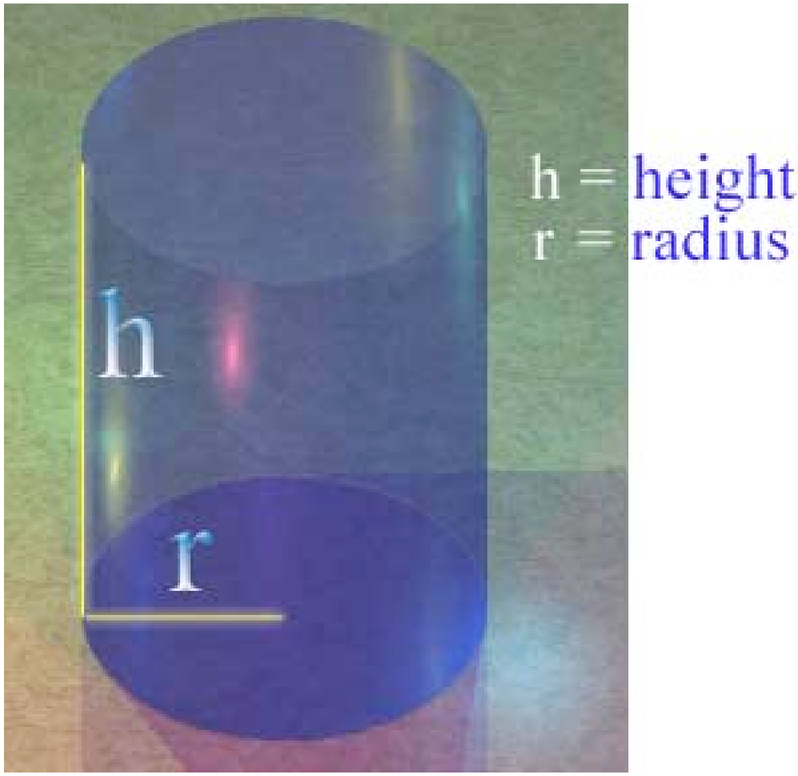
Simple Example (Cylindrical)

- Find the volume *V* of a cylinder radius *r*, height *h*
- In Cartesians:

$$4 \times \int_{z=0}^{z=h} \int_{y=0}^{y=r} \int_{x=0}^{x=(r^2-y^2)^{1/2}} \mathrm{d}x\mathrm{d}y\mathrm{d}z$$

$$= 4h \int_{y=0}^{y=r} (r^2 - y^2)^{1/2} \mathrm{d}y$$

$$= \pi r^2 h$$



- In Cylindrical Polars

$$\int_{z=0}^{z=h} \int_{\phi=0}^{\phi=2\pi} \int_{\rho=0}^{\rho=r} \rho \mathrm{d}\rho \mathrm{d}\phi \mathrm{d}z$$

$$= \pi r^2 h$$

Harder Example (Cylindrical)

- Find the moment of inertia about the *z* axis of the solid that lies below the paraboloid

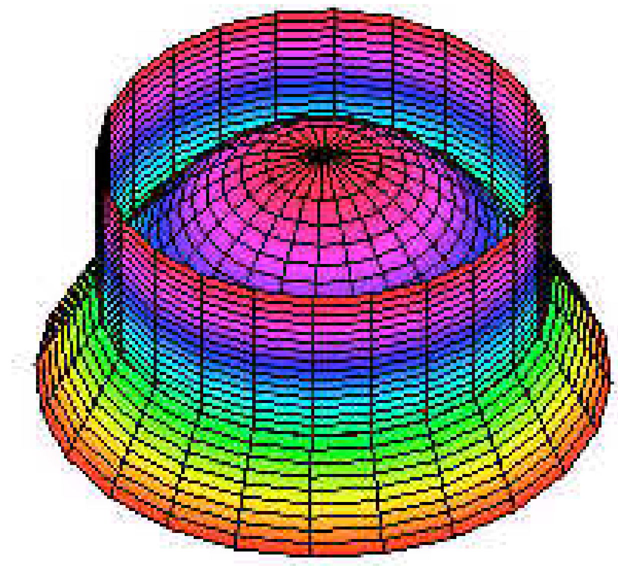
$$z = 25 - x^2 - y^2$$

inside the cylinder

$$x^2 + y^2 = 4$$

and above the *xy* plane, having a density function

$$\rho(x, y, z) = x^2 + y^2 + 6z$$



$$I_z = \int\int\int (x^2 + y^2)(x^2 + y^2 + 6z)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z$$

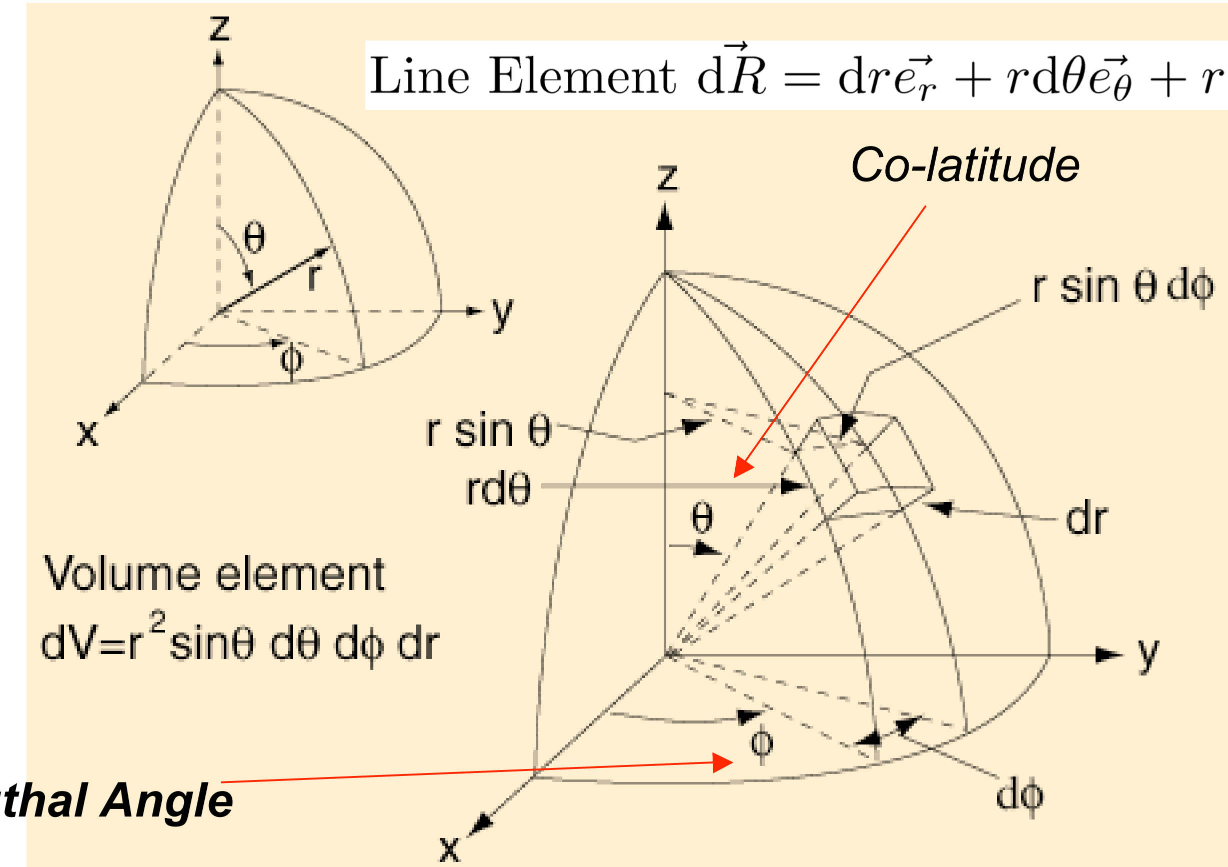
$$\Rightarrow$$

$$I_z = \int_0^{2\pi}\int_0^{2}\int_0^{25-r^2} r^2(r^2+6z)r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^{2}\left[r^5 z + 3r^3 z^2\right]_0^{25-r^2} dr\,d\theta$$

$$\int_0^{2\pi}\int_0^{2}\left(-125r^5 + 2r^7 + 1875r^3\right)dr\,d\theta = \frac{37384\pi}{3}$$

Spherical Polar Coordinates

Line Element $\mathrm{d}\vec{R} = \mathrm{d}r\vec{e_r} + r\mathrm{d}\theta\vec{e_\theta} + r\sin\theta\mathrm{d}\theta\vec{e_\phi}$



Azimuthal Angle

"longitude eastwards"

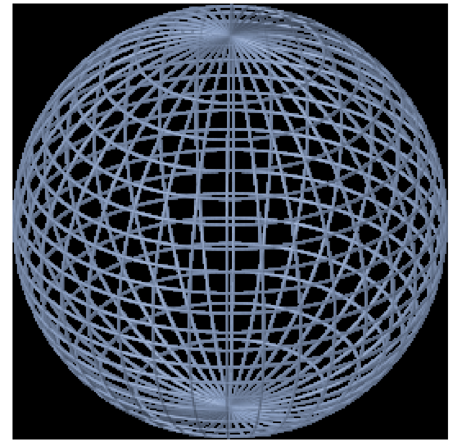
Note *r* is now a radial distance from origin

$$z = r\cos\theta$$
$$x = r\sin\theta\cos\phi$$
$$y = r\sin\theta\sin\phi$$

$$r = (x^2 + y^2 + z^2)^{1/2}$$
$$\tan\phi = \frac{y}{x}$$
$$\tan\theta = \frac{(x^2 + y^2)^{1/2}}{z}$$

Simple Example (Spherical)

- Find the volume *V* of a sphere radius *r*



- In Cartesians:

$$4 \times \int_{x=0}^{x=r} \int_{y=0}^{y=(r^2-x^2)^{1/2}} \int_{z=0}^{z=(r^2-x^2-y^2)^{1/2}} \mathrm{d}x\mathrm{d}y\mathrm{d}z$$

$$= 4 \times \int_{x=0}^{x=r} \int_{y=0}^{y=(r^2-x^2)^{1/2}} (r^2 - x^2 - y^2)^{1/2}\mathrm{d}x\mathrm{d}y$$

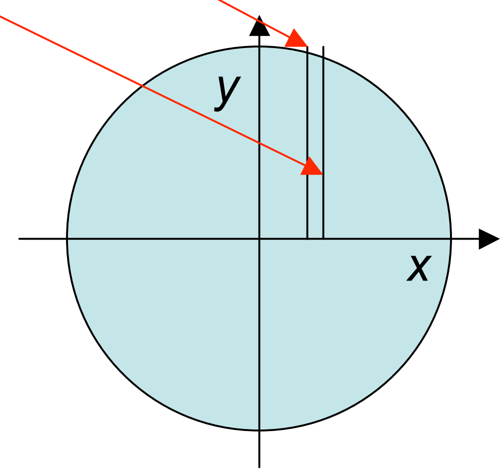
$$= \frac{4\pi r^3}{3}$$

Exercise for student!

Projection onto *z*=0 plane

Height of this strip is

$$z = (r^2 - x^2 - y^2)^{1/2}$$



- In Spherical Polars

$$\int_{\theta=0}^{\theta=\pi} \int_{\phi=0}^{\phi=2\pi} \int_{\rho=0}^{\rho=r} \rho^2 \sin\theta \mathrm{d}\rho\mathrm{d}\phi\mathrm{d}\theta$$

$$= \frac{4\pi r^3}{3}$$

Harder Example (Spherical)

- Find the volume V that lies inside the sphere

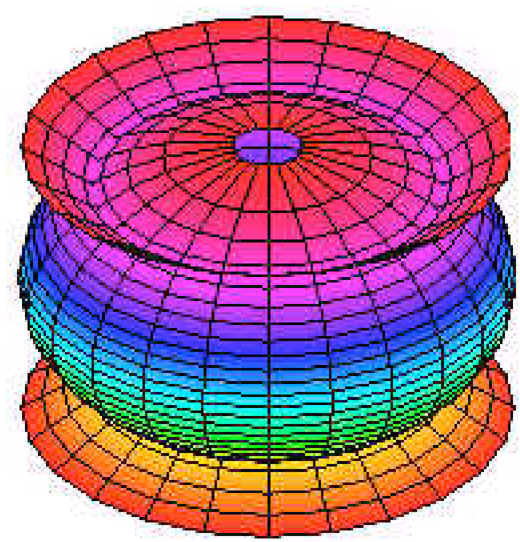
$$x^2 + y^2 + z^2 = 2$$

and outside the cone

$$z^2 = x^2 + y^2$$

- In spherical polars

$$r = \sqrt{2} \text{ and } \cos^2\theta = \frac{1}{2}$$



$$\Rightarrow \quad V = \int_0^{2\pi} \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \int_0^{\sqrt{2}} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta \qquad V = \frac{8\pi}{3}$$