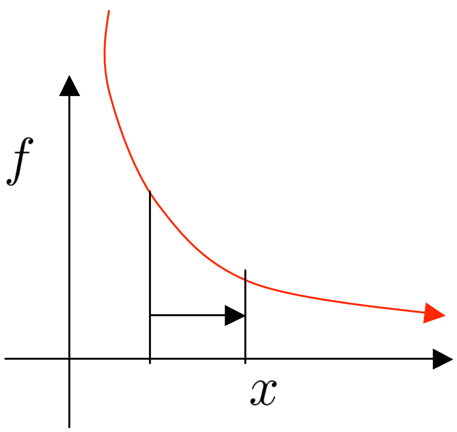
Lecture 5: Jacobians

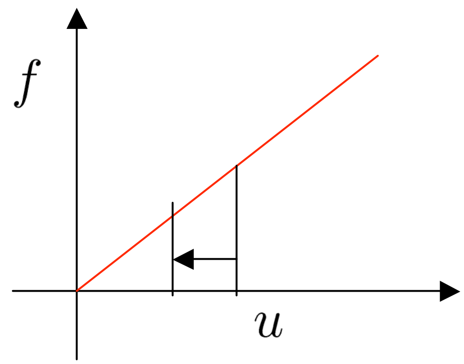
- In 1D problems we are used to a simple change of variables, e.g. from *x* to *u*

$$\int_a^b f(x)\,\mathrm{d}x = \int_\alpha^\beta f(x(u))\,\boxed{\frac{\mathrm{d}x}{\mathrm{d}u}}\,\mathrm{d}u$$

1D Jacobian

maps strips of width d*x* to strips of width d*u*





- Example: $\int_1^2 \frac{1}{x}\mathrm{d}x = \ln(2)$ Substitute $x = u^{-1} \longrightarrow \frac{\mathrm{d}x}{\mathrm{d}u} = -u^{-2}$

$$= -\int_1^{\frac{1}{2}} \frac{u}{u^2}\mathrm{d}u = [\ln u]_{\frac{1}{2}}^{1} = \ln(2)$$

2D Jacobian

- For a continuous 1-to-1 transformation from (*x,y*) to (*u,v*)
- Then $$x = x(u,v) \text{ and } y = y(u,v)$$

$$\int\int_R f(x,y)\mathrm{d}x\mathrm{d}y = \int\int_{R'} f(x(u,v),y(u,v)) \boxed{\left|\frac{\partial(x,y)}{\partial(u,v)}\right|} \mathrm{d}u\mathrm{d}v$$

- Where Region (in the *xy* plane) maps onto region R in the *uv* plane R'

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

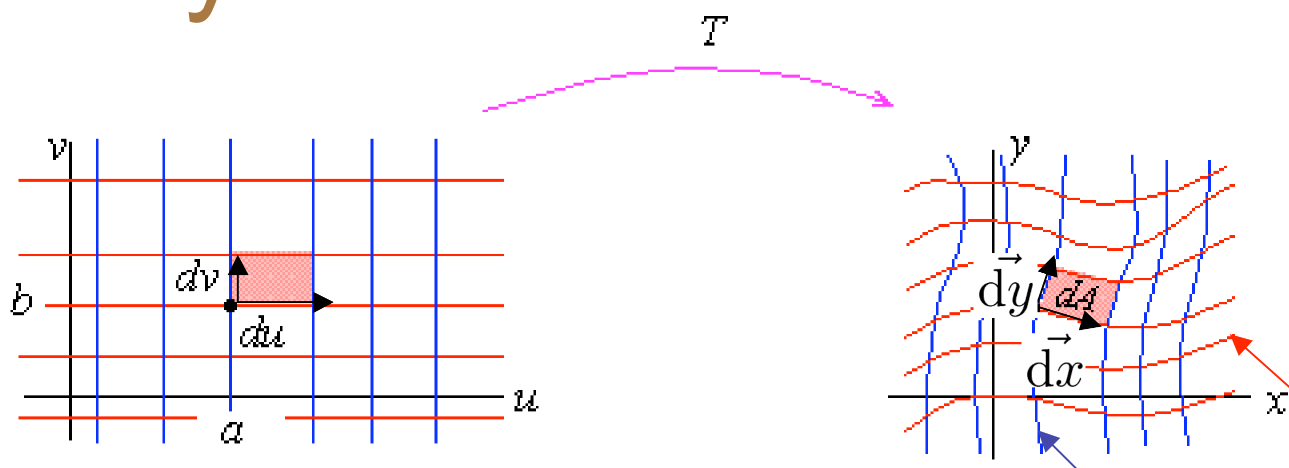
2D Jacobian

maps areas d*x*d*y* to areas d*u*d*v*

- Hereafter call such terms x_u

$$= \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = x_u y_v - x_v y_u$$

Why the 2D Jacobian works



- Transformation *T* yield distorted grid of lines of constant *u* and constant *v*
- For small d*u* and d*v,* rectangles map onto parallelograms

$$\vec{\mathrm{d}u} = (\mathrm{d}u, 0) \quad \text{and} \quad \vec{\mathrm{d}v} = (0, \mathrm{d}v)$$

$$\vec{\mathrm{d}x} = \begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix} \begin{pmatrix} \mathrm{d}u \\ 0 \end{pmatrix} \qquad \vec{\mathrm{d}y} = \begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix} \begin{pmatrix} 0 \\ \mathrm{d}v \end{pmatrix}$$

$$\mathrm{d}A = |\vec{\mathrm{d}x} \times \vec{\mathrm{d}y}| = \begin{pmatrix} x_u \mathrm{d}u \\ y_u \mathrm{d}u \end{pmatrix} \times \begin{pmatrix} x_v \mathrm{d}v \\ y_v \mathrm{d}v \end{pmatrix} = (x_u y_v - x_v y_u)\mathrm{d}u\mathrm{d}v$$

- This is a ***Jacobian***, i.e. the determinant of the ***Jacobian Matrix***

Relation between Jacobians

- The Jacobian matrix $\frac{\partial(x,y)}{\partial(u,v)}$ is the ***inverse matrix*** of $\frac{\partial(u,v)}{\partial(x,y)}$ i.e.,

$$\begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix} \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

- Because (and similarly for d*y)*

$$\mathrm{d}x = x_u \mathrm{d}u + x_v \mathrm{d}v = x_u \mathrm{d}u + x_v (v_x \mathrm{d}x + v_y \mathrm{d}y)$$

$$\begin{array}{ll} x \text{ constant} & \Rightarrow \mathrm{d}x = 0 \Rightarrow 0 = x_u u_y + x_v v_y \\ y \text{ constant} & \Rightarrow \mathrm{d}y = 0 \Rightarrow 1 = x_u u_x + x_v v_x \end{array}$$

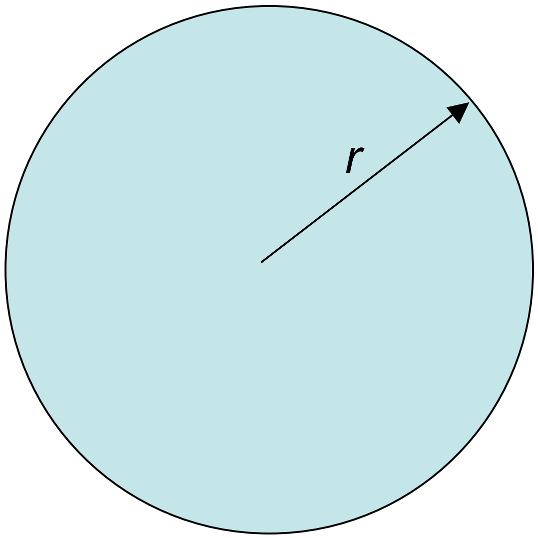
- This makes sense because Jacobians measure the relative areas of d*x*d*y* and d*u*d*v, i.e*

$$\det(AB) = \det(A)\det(B) = 1 \Rightarrow \det(A) = \frac{1}{\det B}$$

- So

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \left|\frac{\partial(u,v)}{\partial(x,y)}\right|^{-1}$$

Simple 2D Example



Area of circle *A*=

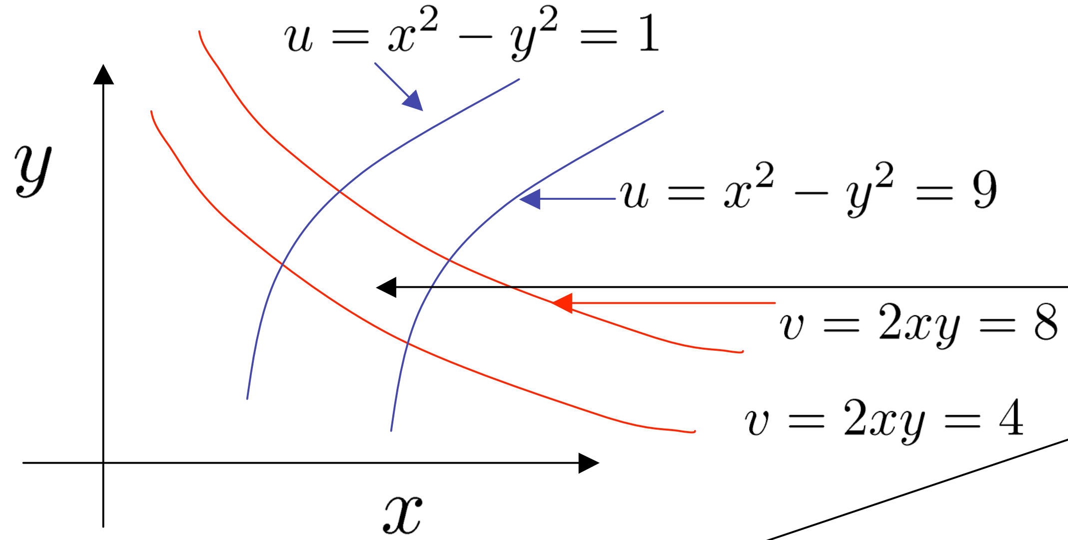
$$\int\int_A \mathrm{d}x\mathrm{d}y$$

$$x = \rho\cos\theta \quad \text{and} \quad y = \rho\sin\theta$$

$$\left| \begin{array}{cc} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \theta} \end{array} \right| = \left| \begin{array}{cc} \cos\theta & -\rho\sin\theta \\ \sin\theta & \rho\cos\theta \end{array} \right| = \rho(\cos^2\theta + \sin^2\theta) = \rho$$

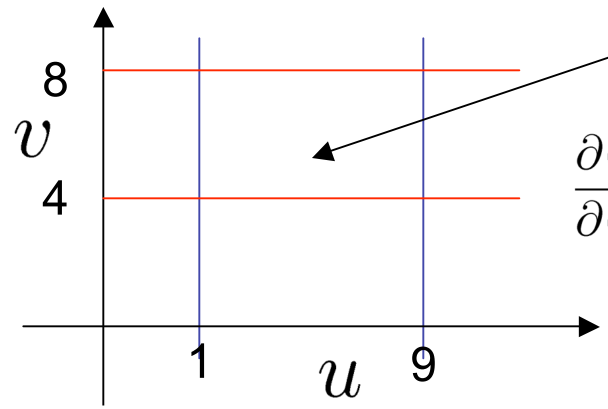
$$A = \int_{\rho=0}^{\rho=r}\int_{\theta=0}^{\theta=2\pi} \rho\mathrm{d}\rho\mathrm{d}\theta = \left[\tfrac{1}{2}\rho^2\right]_0^r [\theta]_0^{2\pi} = \pi r^2$$

Harder 2D Example



$$\int\int_R (x^2 + y^2)\mathrm{d}x\mathrm{d}y$$

where *R* is this region of the *xy* plane, which maps to *R'* here



$$\frac{\partial(x,y)}{\partial(u,v)} = \left[\frac{\partial(u,v)}{\partial(x,y)}\right]^{-1} = \begin{vmatrix} 2x & -2y \\ 2y & 2x \end{vmatrix}^{-1} = \frac{1}{4(x^2+y^2)}$$

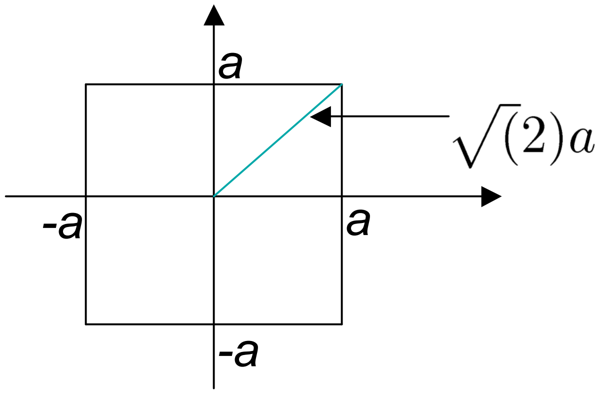
but $u^2 + v^2 = (x^2 - y^2)^2 + 4x^2y^2 = (x^2 + y^2)^2$ so $\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{4\sqrt{(u^2+v^2)}}$

$$\int\int_R (x^2 + y^2)\mathrm{d}x\mathrm{d}y = \frac{1}{4}\int_1^9\int_4^8 \mathrm{d}u\mathrm{d}v = 8$$

An Important 2D Example

- Evaluate

$$I = \int_{-\infty}^{\infty} e^{-x^2} \mathrm{d}x$$



- First consider

$$I_a^2 = \int_{-a}^{a} e^{-x^2} \mathrm{d}x \int_{-a}^{a} e^{-y^2} \mathrm{d}y$$

$$I_a^2 = \int_{-a}^{a} \int_{-a}^{a} e^{-(x^2+y^2)} \mathrm{d}y \mathrm{d}x$$

- Put $x = r\cos\phi$ and $y = r\sin\phi$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \cos\phi & -r\cos\phi \\ \sin\phi & r\cos\phi \end{vmatrix} = r$$

$$\int_0^a \int_0^{2\pi} re^{-r^2} \mathrm{d}r\mathrm{d}\phi < I_a^2 < \int_0^{\sqrt{2}a} \int_0^{2\pi} re^{-r^2} \mathrm{d}r\mathrm{d}\phi$$

- $\pi(1-e^{-a^2}) < I_a^2 < \pi(1-e^{-2a^2})$ as $a \to \infty \Rightarrow I_a = \sqrt{\pi}$

3D Jacobian

$$x = x(u, v, w), y = y(u, v, w), z = z(u, v, w)$$

- maps volumes (consisting of small cubes of volume $\mathrm{d}x\mathrm{d}y\mathrm{d}z$
- ....to small cubes of volume $\mathrm{d}u\mathrm{d}v\mathrm{d}w$

$$\int\int\int_V f(x, y, z)\mathrm{d}x\mathrm{d}y\mathrm{d}z = \int\int\int_{V'} F(u, v, w) \left|\frac{\partial(x, y, z)}{\partial(u, v, w)}\right| \mathrm{d}u\mathrm{d}v\mathrm{d}w$$

- Where

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{vmatrix}$$

3D Example

- Transformation of volume elements between Cartesian and spherical polar coordinate systems (see Lecture 4)

$$\begin{aligned} x &= r\sin\theta\cos\phi \\ y &= r\sin\theta\sin\phi \\ z &= r\cos\theta \end{aligned}$$

$$\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

$$= r^2\sin\theta$$