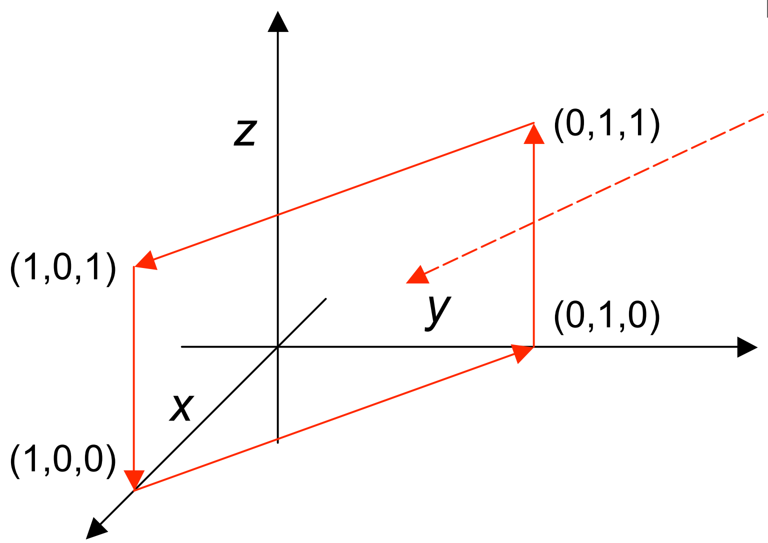
Lecture 6: Surface Integrals

- Recall, area is a vector

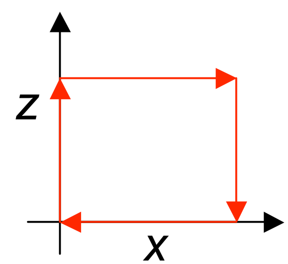


- Vector area of this surface is

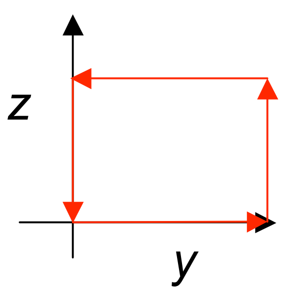
$$[(0,1,0)-(1,0,0)]\times[(1,0,1)-(1,0,0)]$$

$$=(1,1,0)$$ which has sensible magnitude $\sqrt{2}$ and direction

- Or, by projection



$$S_y=+1$$



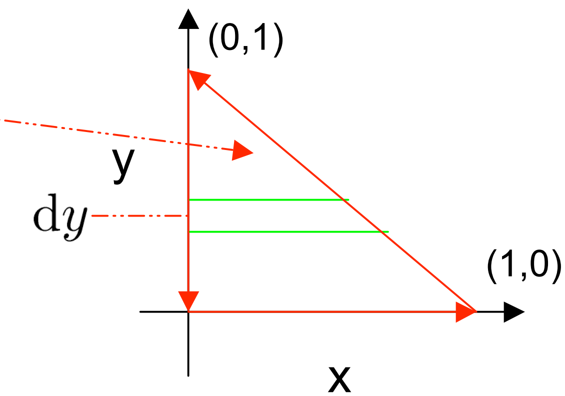
$$S_x=+1$$

Surface Integrals

- Example from Lecture 3 of a ***scalar field*** integrated over a small (differential) ***vector*** surface element $\mathrm{d}\vec{S}$ in plane z=0

- Example

$$\int\int_S x^2 y \ \mathrm{d}x\mathrm{d}y \ \text{over}$$



Do *x* first

$$\mathrm{d}y\left(\int_0^{1-y} x^2 y \ \mathrm{d}x\right) = \frac{y\mathrm{d}y}{3}\left[x^3\right]_0^{1-y} = \frac{y\mathrm{d}y}{3}(1-y)^3$$

Then do *y*

$$\int_0^1 \left(\frac{y}{3} - y^2 + y^3 - \frac{y^4}{3}\right)\mathrm{d}y = \frac{1}{60}$$

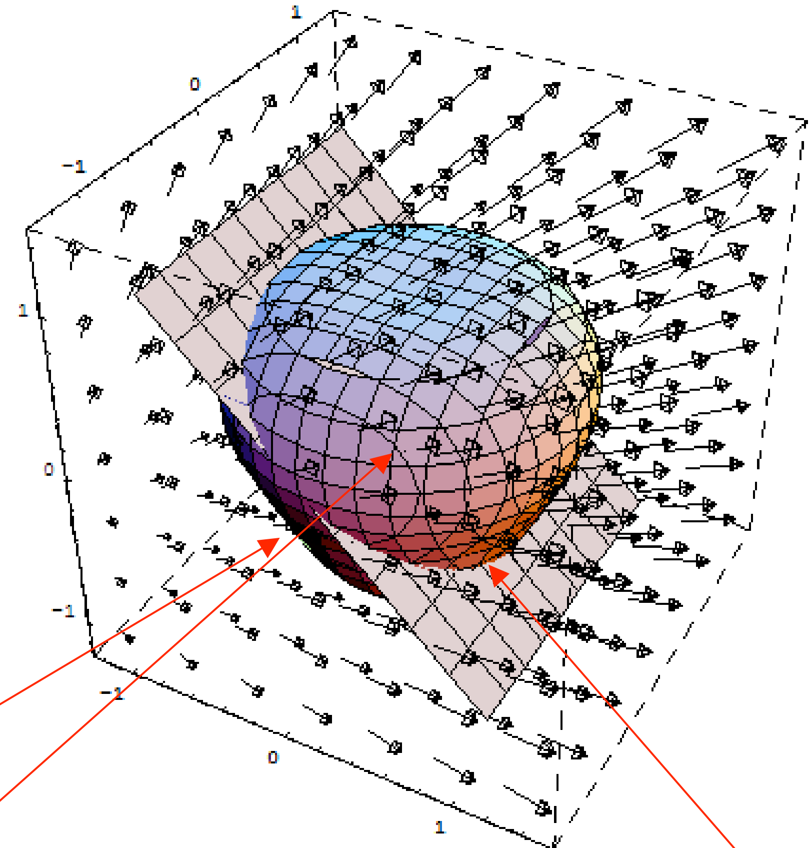
- But answer is a vector $= (0, 0, +\frac{1}{60})$; get positive sign by applying right-hand rule to an assumed path around the edge (z>0 is "outside" of surface)

Vector Surface Integrals

Example: volume flow of fluid, with velocity field $\vec{V}(\vec{r})$, through a surface element $\mathrm{d}\vec{S}$ yielding a scale quantity

$$\text{Rate} \;= \mathrm{d}\vec{S}\cdot\vec{V}$$

$$\int\int_S \vec{V}.\mathrm{d}\vec{S} = \lim_{\delta S\to 0}\left[\sum_i V_i \mathrm{d}S_i \cos\theta_i\right]$$



Consider an ***incompressible fluid***

"What goes into surface S through S_1 comes out through surface S_2"

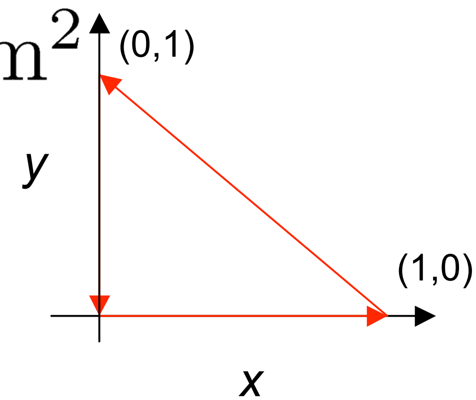
$$\int\int_{S_1} \vec{V}.\mathrm{d}\vec{S} = -\int\int_{S_2} \vec{V}.\mathrm{d}\vec{S} \to \int\int_S \vec{V}.\mathrm{d}\vec{S} = 0$$

and depends only on rim not surface

S_1 and S_2 end on same ***rim***

Examples

- Consider a simple fluid velocity field $\vec{v} = (0, 0, 1)\ \mathrm{ms}^{-1}$
- Through the triangular surface $\vec{S} = (0, 0, +\frac{1}{2})\ \mathrm{m}^2$



$$\int\int \vec{v}\cdot \mathrm{d}\vec{S} = \vec{v}\cdot\vec{S} = \frac{1}{2}\ \mathrm{m}^3\mathrm{s}^{-1}$$

- More complicated fluid flow requires an integral $\vec{v} = (xz, z^2, x^2 y)\ \mathrm{ms}^{-1}$

$$\int\int \vec{v}\cdot \mathrm{d}\vec{S} = \int\int (x^2 y)\mathrm{d}x\mathrm{d}y = \frac{1}{60}\ \mathrm{m}^3\mathrm{s}^{-1}$$

(see second sheet of this Lecture)

Evaluation of Surface Integrals by Projection

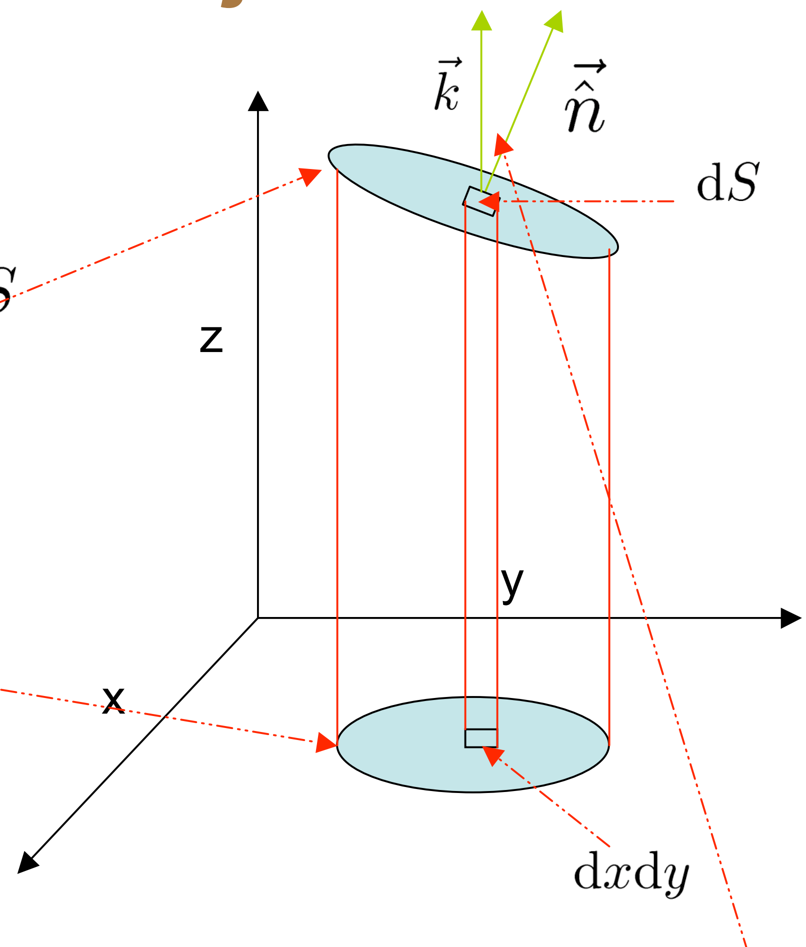
want to calculate

$$\int\int_S \vec{A}.\mathrm{d}\vec{S} = \int\int_S \vec{A}.\vec{\hat{n}}\mathrm{d}S$$

$$= \int\int_R \vec{A}.\vec{\hat{n}}\frac{\mathrm{d}x\mathrm{d}y}{|\vec{n}.\vec{k}|}$$

because

$$\mathrm{d}S\cos\theta = \mathrm{d}x\mathrm{d}y$$



Note S need not be planar!

Note also, project onto easiest plane

$$\vec{\hat{n}}.\vec{k} = |\vec{\hat{n}}||\vec{k}|\cos\theta = \cos\theta$$

Example

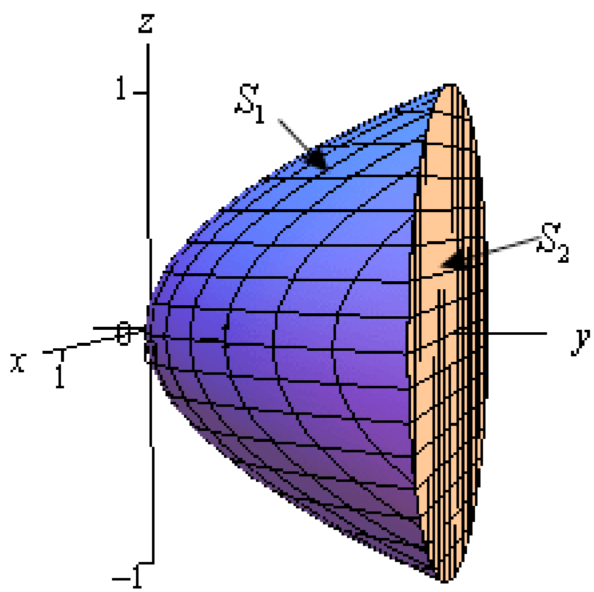
- Surface Area of some general shape $z = f(x, y)$

$$= \int \int_R \vec{A}.\vec{\hat{n}} \frac{\mathrm{d}x\mathrm{d}y}{|\vec{n}.\vec{k}|} \quad \text{where } \vec{A} = \vec{\hat{n}}$$

$$\vec{\hat{n}} = \frac{(f_x, f_y, -1)}{\sqrt{1+f_x^2+f_y^2}}$$

$$= \int \int_R \sqrt{1 + f_x^2 + f_y^2}\mathrm{d}x\mathrm{d}y$$

Another Example



$$\vec{F} = (0, y, -z)$$

- and S is the closed surface given by the paraboloid S_1 $\quad y = x^2 + z^2$

  and the disk S_2 $\quad x^2 + z^2 < 1 \text{ at } y = 1$

- On S_1 : $\quad \vec{\hat{n}} = \dfrac{(2x,-1,2z)}{\sqrt{1+4x^2+4z^2}}$

$$\int\int_{s_1} \vec{F} \cdot \mathrm{d}\vec{S} = \int\int \mathrm{d}x\mathrm{d}z(-y - 2z^2) = \int\int \mathrm{d}x\mathrm{d}z(-x^2 - 3z^2) = -\pi$$

Exercises for student (in polars over the unit circle, i.e. projection of both S_1 and S_2 onto xz plane)

- On S_2 : $\int\int_{S_2} \vec{F} \cdot \mathrm{d}\vec{S} = \int\int y\mathrm{d}x\mathrm{d}z = \pi$
- So total "flux through closed surface" (S_1+S_2) is zero