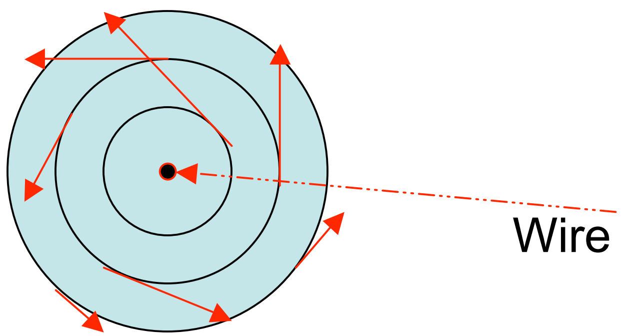
Lecture 7: Non-conservative Fields and The Del Operator

- mathematical vector fields, like (y^3, x) from Lecture 3, are ***non conservative***
- so is ***magnetic field*** $\vec{B}$ $\quad \mathrm{since\ Ampere'sLaw} \rightarrow \oint \vec{B}.\vec{\mathrm{dl}} = \mu_0 I \neq 0$

"closed-loop" integral is non-zero



We will learn later that ***multi-valued*** scalar potentials can be used for such fields.

Non Conservative Fields

INTEGRATING FACTOR

Turns a ***non-conservative*** vector field into a ***conservative*** vector field.

Example $$\vec{A}.\mathrm{d}\vec{l} \text{ with } \vec{A} = (-y, x) \text{ and } \mathrm{d}\vec{l} = (\mathrm{d}x, \mathrm{d}y)$$

$$đ\psi = \vec{A}.\mathrm{d}\vec{l} = -y\mathrm{d}x + x\mathrm{d}y$$ is ***inexact*** because if it were ***exact***

$$\mathrm{d}\psi = \frac{\partial\psi}{\partial x}.\mathrm{d}x + \frac{\partial\psi}{\partial y}.\mathrm{d}y$$ and hence $$-y = \frac{\partial\psi}{\partial x} \xrightarrow{\int} \psi = -yx + C(y)$$

$$x = \frac{\partial\psi}{\partial y} \xrightarrow{\int} \psi = xy + D(x)$$

These equations cannot be made consistent for any ***arbitrary functions*** C and D.

Integrability Condition

General differential

$$\mathrm{d}\phi = P(x,y)\mathrm{d}x + Q(x,y)\mathrm{d}y$$

is integrable if

$$P = \frac{\partial \phi}{\partial x}; \qquad Q = \frac{\partial \phi}{\partial y}$$

partially differentiate w.r.t. y

partially differentiate w.r.t. x

$$\frac{\partial P}{\partial y} = \frac{\partial^2 \phi}{\partial y \partial x} \qquad \frac{\partial Q}{\partial x} = \frac{\partial^2 \phi}{\partial x \partial y}$$

equal for all well-behaved functions

Or integrability condition $$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

In previous example $P = -y \rightarrow \frac{\partial P}{\partial y} = -1 \quad Q = x \rightarrow \frac{\partial Q}{\partial x} = 1$

Since these are NOT the same, ***not integrable***

Example Integrating Factor

- often, inexact differentials can be made exact with an integrating factor
- Example

$$\frac{1}{x^2+y^2}\,\text{đ}\psi = \frac{-y}{(x^2+y^2)}\,\mathrm{d}x + \frac{x}{(x^2+y^2)}\,\mathrm{d}y$$

- Now

$$\frac{\partial P}{\partial y} = \frac{-(x^2+y^2)+2y^2}{(x^2+y^2)^2}$$

$$\frac{\partial Q}{\partial x} = \frac{(x^2+y^2)-2x^2}{(x^2+y^2)^2}$$

are now equal

$$\mathrm{d}\phi = \frac{\text{đ}\psi}{(x^2+y^2)}$$

defines a potential, or ***state***, function $\phi(x,y)$

Conservative Fields

In a Conservative Vector Field $\vec{A}$

$$\int_1^2 \vec{A}.\vec{\mathrm{d}l} \text{ is independent of path } \equiv \text{vector field } \vec{A} = \vec{\nabla}\phi \text{ for scalar potential } \phi$$

$$\mathrm{d}\phi = \frac{\partial \phi}{\partial x}\mathrm{d}x + \frac{\partial \phi}{\partial y}\mathrm{d}y + \frac{\partial \phi}{\partial z}\mathrm{d}z = \vec{\nabla}\phi.\vec{\mathrm{d}l}$$

$$\int_1^2 \vec{A}.\vec{\mathrm{d}l} = \mathrm{d}\phi = \phi_2 - \phi_1$$

Which gives an easy way of evaluating line integrals: regardless of path, it is difference of potentials at points 1 and 2.

$$\oint \vec{A}.\vec{\mathrm{d}l} = 0$$

Obvious provided potential is single-valued at the start and end point of the closed loop.

Del

$$\vec{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$$

Grad

$$\vec{\nabla}\phi = \vec{A}$$

"Vector operator acts on a scalar field to generate a vector field"

Example

$$\vec{\nabla}(2x + 3y^2 - \sin z) = (2, 6y, -\cos z)$$

Div

$$\vec{\nabla} \cdot \vec{A} = \psi$$

"Vector operator acts on a vector field to generate a scalar field"

Example

$$\vec{\nabla} \cdot (xy, y^2 z, \sin z) = y + 2yz + \cos z$$

Curl

$$\vec{\nabla} \times \vec{A} = \vec{B}$$

"Vector operator acts on a vector field to generate a vector field"

Example

$$\vec{\nabla} \times (xy, y^2 z, \sin z) = (-y^2, 0, -x)$$