Lecture 8: Div(ergence)

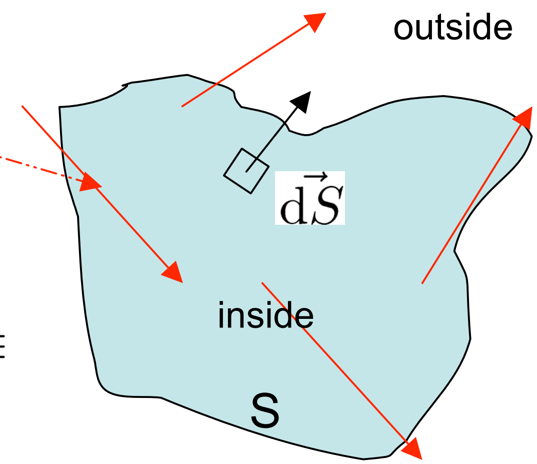
- Consider vector field $\vec{A}$ and

$\oint_S \vec{A}.\mathrm{d}\vec{S}$ over a closed surface $\equiv$

Flux (c.f. fluid flow of Lecture 6) of $\vec{A}$ out of *S*

e.g, if $\vec{A}$ is the fluid velocity (in m s^{-1}), $\rho \int \vec{A}.\mathrm{d}\vec{S} \equiv$

rate of flow of material (in kg s^{-1}) ***out of*** S



- For many vector fields, e.g. ***incompressible*** fluid velocity fields, constant fields and magnetic fields $\oint \vec{A} \cdot \mathrm{d}\vec{S} = 0$

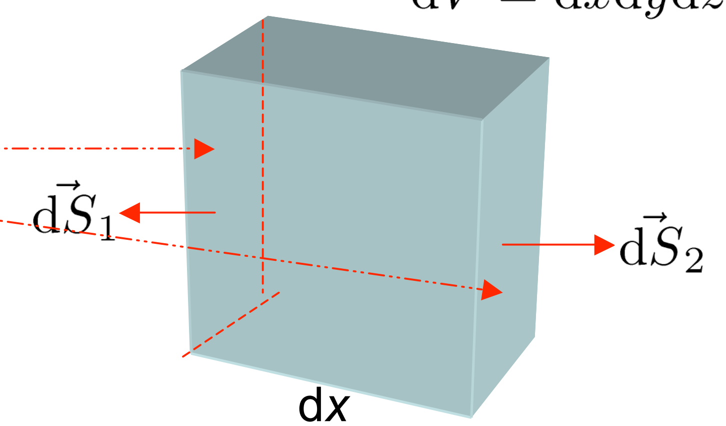
- But sometimes $\oint \vec{A} \cdot \mathrm{d}\vec{S} \neq 0$ and we define div(ergence) for these cases by

$$\mathrm{div}\vec{A} \equiv \lim_{\mathrm{d}V \to 0} \frac{\oint \vec{A} \cdot \mathrm{d}\vec{S}}{\mathrm{d}V}$$

A ***scalar*** giving flux/unit volume (in s^{-1}) out of $\mathrm{d}V$

In Cartesians

- Consider tiny volume with sides d*x*, d*y*, d*z* so that $\vec{A}$ varies linearly across it



- Consider opposite areas 1 and 2

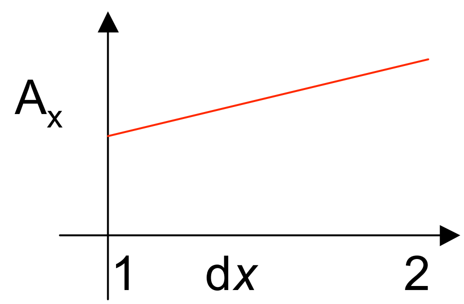
$\int \vec{A} \cdot \mathrm{d}\vec{S}$ for just these two areas =

$$-A_x|_1 \mathrm{d}S + A_x|_2 \mathrm{d}S$$

$$\text{where } \mathrm{d}\vec{S}_2 = (\mathrm{d}S, 0, 0) = (\mathrm{d}y\mathrm{d}z, 0, 0)$$

$$\text{and } A_x|_2 - A_x|_1 = \frac{\partial A_x}{\partial x}\mathrm{d}x$$

$$\text{total flux } = \frac{\partial A_x}{\partial x}.\mathrm{d}x\mathrm{d}S = \frac{\partial A_x}{\partial x}\mathrm{d}V$$



- Similar contributions from other pairs of surfaces

$$\Rightarrow \mathrm{div}\vec{A} = \lim_{\mathrm{d}V \to 0} \frac{\left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right)\mathrm{d}V}{\mathrm{d}V} \Rightarrow$$

$$\mathrm{div}\vec{A} = \vec{\nabla}.\vec{A}$$

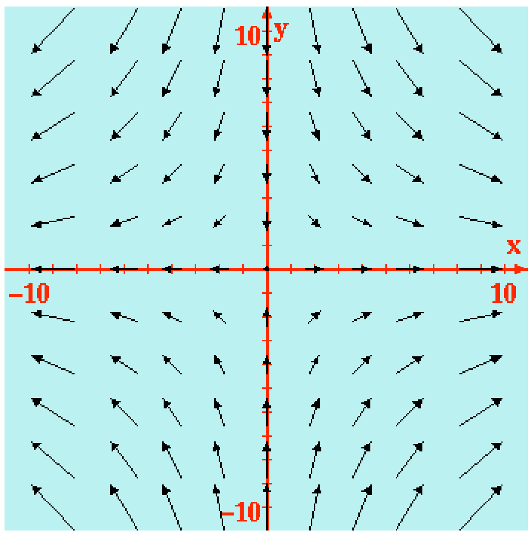
Simple Examples

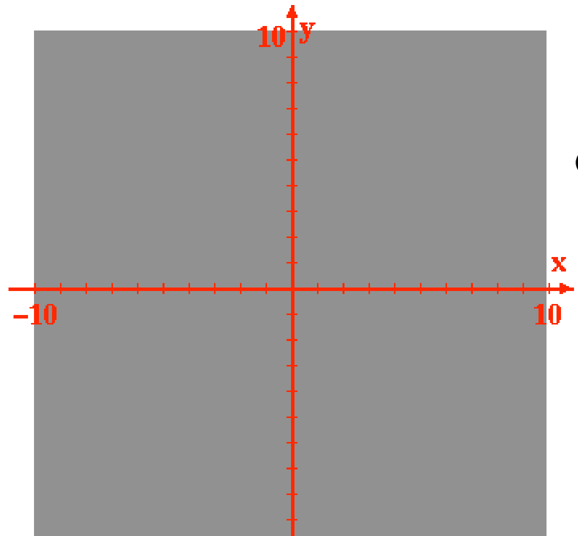
$$\vec{\nabla}.\vec{r} = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}).(x, y, z) = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 3$$

$$\begin{aligned} \vec{\nabla}.(y^2 z, xy, \sin z) &= \\ \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right).(y^2 z, xy, \sin z) &= \\ \frac{\partial(y^2 z)}{\partial x} + \frac{\partial(xy)}{\partial y} + \frac{\partial \sin z}{\partial z} &= \\ 0 + x + \cos z &= \quad x + \cos z \end{aligned}$$

Graphical Representation

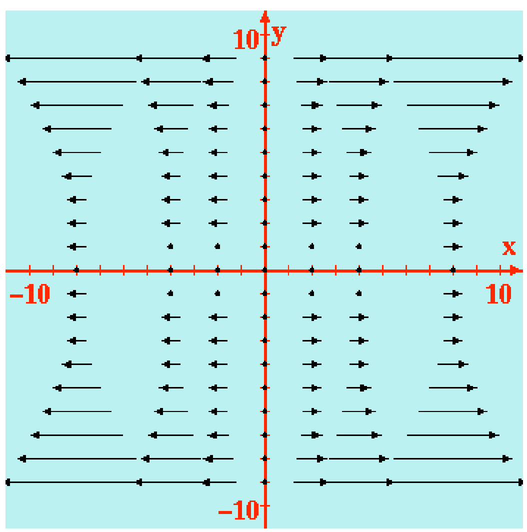
$\vec{A} = (x, -y, 0)$

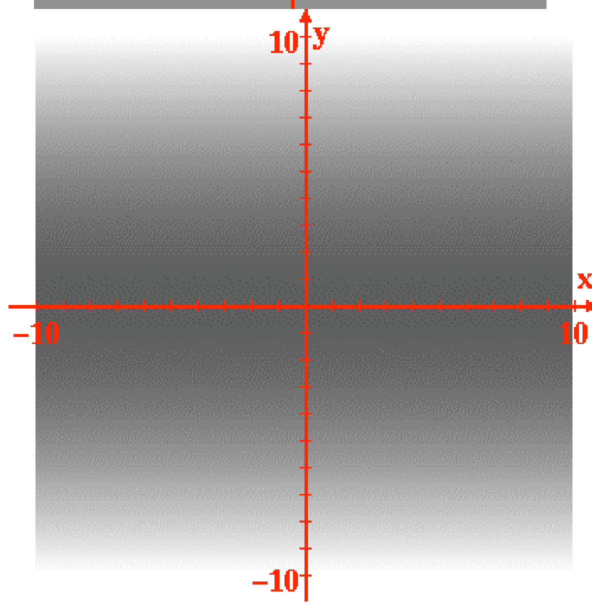




$\vec{\nabla}.\vec{A} = 0$
everywhere

$\vec{A} = (xy^2, 0, 0)$





$\vec{\nabla}.\vec{A} = y^2$

where brightness
encodes value of
scalar function

Physics Examples

- Incompressible (ρ = constant) fluid with velocity field $\vec{V}$

$$\rho \oint_S \vec{V}.\mathrm{d}\vec{S} = 0$$

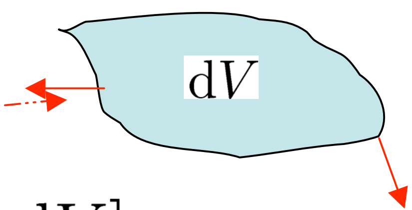
because for any closed surface as much fluid flows out as flows in

$$\Rightarrow \vec{\nabla}.\vec{V} = 0 \text{ everywhere}$$

- Constant field $\oint \vec{A}.\mathrm{d}\vec{S} = \vec{A}.\oint \mathrm{d}\vec{S} = \vec{A}.\vec{S} = 0$

since for any closed surface $\vec{S} = (0,0,0)$

- Compressible ($\rho \neq$ constant) fluid: ρ can change if material flows out/in of d*V*



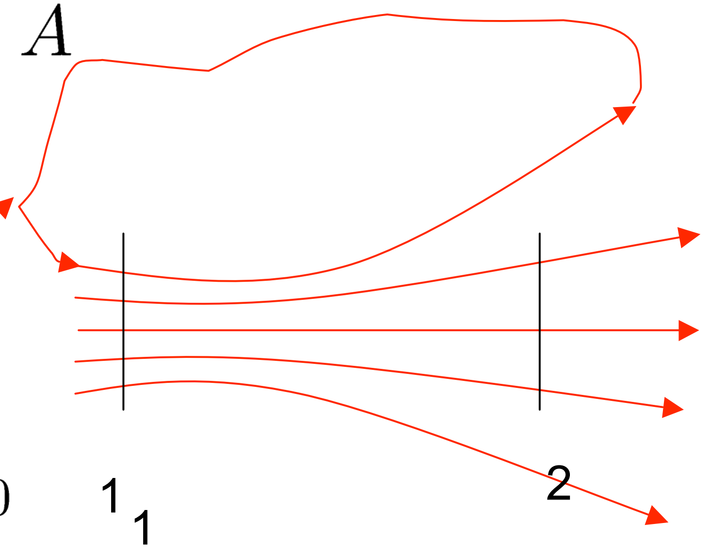
$\rho\vec{V}.\mathrm{d}\vec{S} \equiv$ mass leaving d*V* *per* unit time $\equiv -\frac{\partial}{\partial t}[\rho \mathrm{d}V]$

$\Rightarrow -\frac{\partial \rho}{\partial t} = \mathrm{div}(\rho\vec{V}) = \vec{\nabla} \cdot (\rho\vec{v})$ A ***continuity equation***: "density drops if stuff flows out"

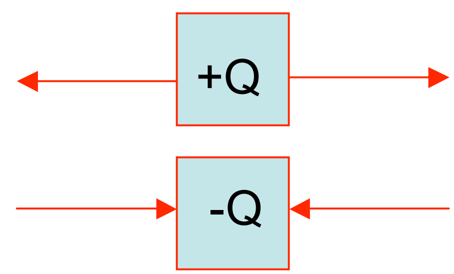
Lines of Force

- Sometimes represent vector fields by ***lines of force***
- Direction of lines is direction of $\vec{A}$
- Density of lines measures $|\vec{A}|$

- Note that div ($\vec{A}$) is the rate of creation of field lines, not whether they are diverging
- If these field lines all ***close***, then $\vec{\nabla}.\vec{A} = 0$
- e.g. for magnetic field lines, there are no ***magnetic monopoles*** that could create field lines $\Rightarrow \vec{\nabla}.\vec{B} = 0 \text{ everywhere}$
- Electric field lines begin and end on ***charges***:
  sources in dV yield positive divergence
  sinks in dV yield negative divergence



$|\vec{A}|$ larger at 1 than 2



Further Physics Examples

For electric charge continuity equation is $-\frac{\partial}{\partial t}\left[\rho_f \mathrm{d}V\right] = \vec{j}.\mathrm{d}\vec{S}$

(s^{-1}) ***charge density*** (C m^{-3}) volume (m^3) ***current density*** (A m^{-2}) surface (m^2)

$\rho_f \equiv n_\mathrm{e} e$ $\qquad$ $n_\mathrm{e}\vec{v}e$

number density (m^{-3}) of charges

So $\quad \mathrm{div}\vec{j} = -\frac{\partial \rho_f}{\partial t}$

drift velocity field charge of current carrier

- "Charge density drops if charges flow out of d*V*" (continuity equation)
- ***Maxwell's Equations*** link properties of ***particles*** (charge) to properties of ***fields*** $\vec{E}$ and $\vec{B}$
- In ***quantum mechanics,*** similar continuity equations emerge from the ***Schrödinger Equation*** in which ρ becomes the **probability density** of a particle being in a given volume and $\vec{j}$ becomes a ***probability current***