Lecture 9: Divergence Theorem

- Consider migrating wildebeest, with velocity vector field

$$\vec{V}(\vec{r}) = (1,1,0)\ \mathrm{m/s}$$
$$\rho = 16\ \mathrm{m}^{-2}$$

- Zero ***influx*** (s^{-1}) into blue triangle:

0 along hypotenuse

$+16\frac{\sqrt{2}}{\sqrt{2}}$ along this side

$-16\frac{\sqrt{2}}{\sqrt{2}}$ along this side

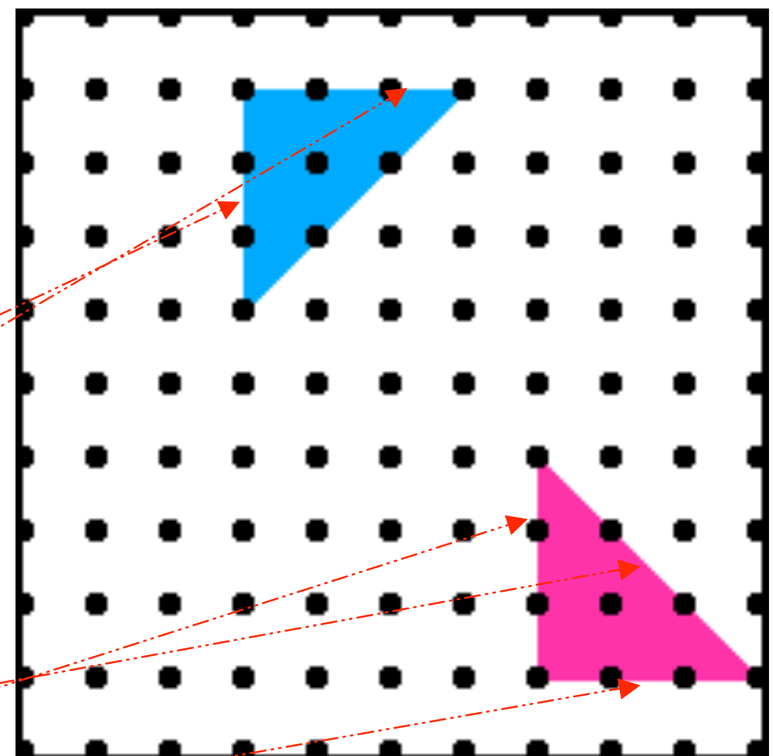
- Zero influx into pink triangle

$+16\frac{\sqrt{2}}{\sqrt{2}}$ along this side

$+16\frac{\sqrt{2}}{\sqrt{2}}$ along this side

$-16\sqrt{2}\sqrt{2}$ along this side

y



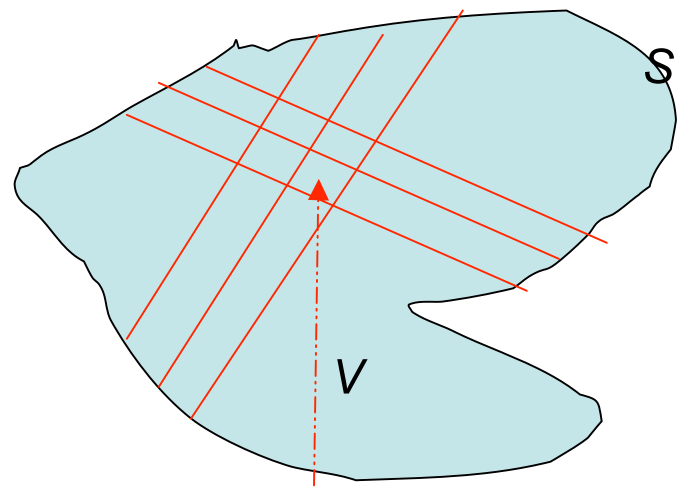
x

because $\vec{\nabla} \cdot \vec{V} = 0$

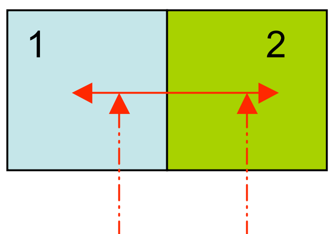
1 m

Proof of Divergence Theorem

- Take definition of div from infinitessimal volume d*V* to ***macroscopic volume*** *V*
- Sum $\int\int \vec{A}.\mathrm{d}\vec{S}$ over all surfaces
- Interior surfaces cancel since $\vec{A}$ is same but vector areas oppose



Divide into infinite number of small cubes dV_i



$$\mathrm{d}\vec{S}_2 = -\mathrm{d}\vec{S}_1$$

DIVERGENCE THEOREM

$$\Rightarrow \oint\oint_S \vec{A}\cdot\mathrm{d}\vec{S} = \underbrace{\int\int\int_V \vec{\nabla}\cdot\vec{A}\,\mathrm{d}V}$$

over ***closed*** outer surface enclosing *V* summed over all tiny cubes

2D Example

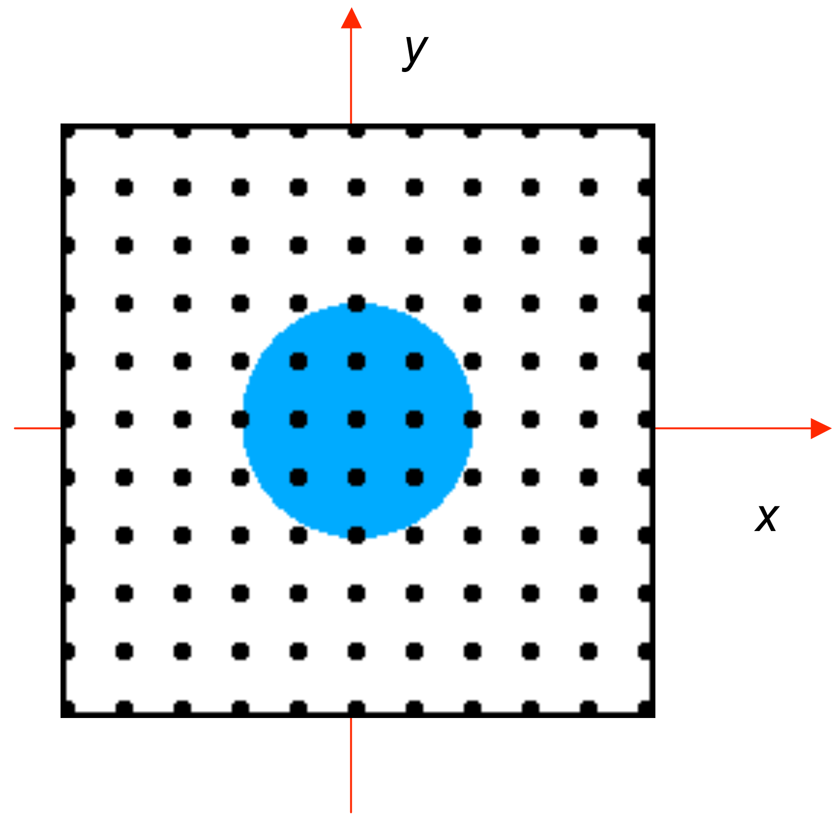
- "outflux (or influx) from a body = total loss (gain) from the ***surface***"

$$\vec{V}(\vec{r}) = (1,1)\ \mathrm{m/s}$$
$$\rho = 16\ \mathrm{m}^{-2}$$

$$\mathrm{d}\vec{S} = \frac{1}{R}(R\cos\theta, R\sin\theta)R\mathrm{d}\theta$$
$$\oint_{\mathrm{perimeter}} \rho\vec{V}.\mathrm{d}\vec{S} =$$
$$\int_0^{2\pi} \rho R(\cos\theta + \sin\theta)\mathrm{d}\theta = 0$$

- exactly as predicted by the divergence theorem since

$$\vec{\nabla}.(\rho\vec{V}) = \rho\vec{\nabla}.\vec{V} = 0 \text{ everywhere}$$
$$\Rightarrow \int\int_{\mathrm{circle}} \vec{\nabla}.(\rho\vec{V})\mathrm{d}V = 0$$



Circle, radius *R* (in m)

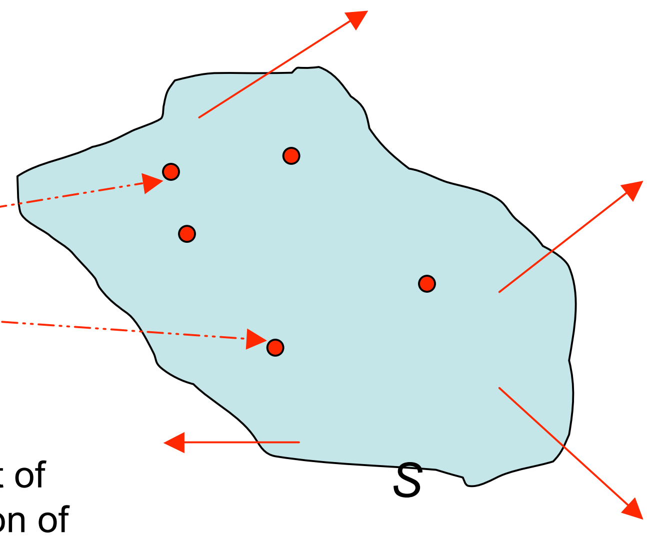
- There is no 'Wildebeest Generating Machine' in the circle!

3D Physics Example: Gauss's Theorem

- Consider tiny spheres around each charge:

$$\oint\oint \vec{E}\cdot \mathrm{d}\vec{S} = \frac{q}{4\pi\epsilon\epsilon_0 r^2}4\pi r^2$$

$$\oint\oint_S \vec{E}\cdot \mathrm{d}\vec{S} = \frac{\sum_i q_i}{\epsilon\epsilon_0}$$



- Where sum is over charges enclosed by *S* in volume *V*

independent of exact location of charges within V

- Applying the ***Divergence Theorem***

$$\oint\oint_S \vec{E}.\mathrm{d}\vec{S} = \int\int\int_V \vec{\nabla}.\vec{E}\mathrm{d}V = \frac{\int\int\int_V \rho_f \mathrm{d}V}{\epsilon\epsilon_0} \Rightarrow \vec{\nabla}.\vec{E} = \frac{\rho_f}{\epsilon\epsilon_0}$$

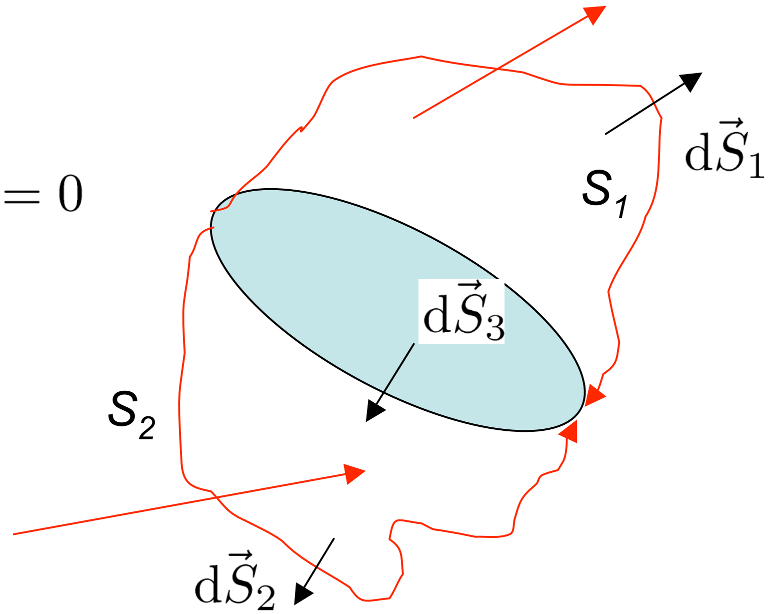
- The ***integral*** and ***differential*** form of ***Gauss's Theorem***

Example: Two non-closed surfaces with same rim

$$\text{if } \mathrm{div}\vec{A} = 0 \text{ then}$$

$$\oint\!\!\oint_S \vec{A}.\mathrm{d}\vec{S} = \int\!\!\int_{S_1} \vec{A}.\mathrm{d}\vec{S} + \int\!\!\int_{S_2} \vec{A}.\mathrm{d}\vec{S} = 0$$

$$\text{and } \int\!\!\int_{S_1} + \int\!\!\int_{S_3} = 0 \text{ and } \int\!\!\int_{S_2} - \int\!\!\int_{S_3} = 0$$



S is ***closed*** surface formed by $S_1 + S_2$

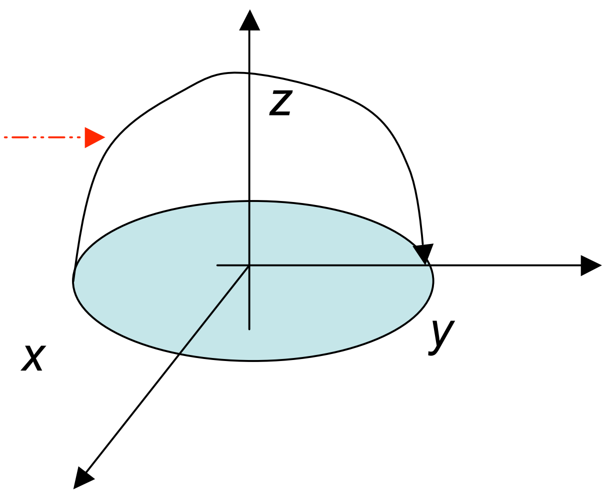
- Which means integral depends on ***rim*** not surface if $\mathrm{div}\vec{A} = 0$
- Special case when $\vec{A} = \mathrm{constant}$ shows why vector area $\vec{S}_1 = -\vec{S}_2$
- If $\mathrm{div}\vec{A} \neq 0$ then integral depends on rim and surface

Divergence Theorem as aid to doing complicated surface integrals

- Example $\int\int_S \vec{F}.\mathrm{d}\vec{S}$ over

open hemisphere $x^2 + y^2 + z^2 = a^2, z \geq 0$

$$\vec{F} = (y - x, x^2 z, z + x^2)$$



- Evaluate Directly $\mathrm{d}\vec{S} = \underbrace{\frac{1}{a}(x, y, z)}_{\hat{\vec{n}}}\ \mathrm{d}S$

on surface $x = a\sin\theta\cos\phi$

$y = a\sin\theta\sin\phi$

$z = a\cos\theta$

$\mathrm{d}S = a^2 \sin\theta \mathrm{d}\theta \mathrm{d}\phi$

- Tedious integral over θ and ϕ (exercise for student!) gives

$$\frac{\pi a^4}{4}$$

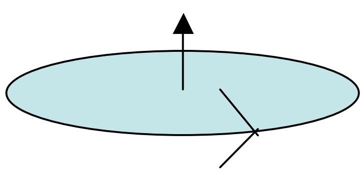
Using the Divergence Theorem

$$\vec{\nabla}.\vec{F} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right).(y - x, x^2 z, z + x^2) = -1 + 0 + 1 = 0$$

- So integral depends only on ***rim***: so do easy integral over circle

$$x^2 + y^2 = a^2, z = 0$$

$$\int\int (y - x, 0, x^2).(0, 0, 1)\mathrm{d}x\,\mathrm{d}y$$



$$= \int_0^a \int_0^{2\pi} r^2 \cos^2\phi\, r\mathrm{d}r\mathrm{d}\phi = \frac{a^4}{4}\int_0^{2\pi} \cos^2\phi\,\mathrm{d}\phi$$

$$= \frac{\pi a^4}{4}$$

- as always beware of signs

$$\int\int_{\text{hemisphere}} - \frac{\pi a^4}{4} = 0 \Rightarrow \int\int_{\text{hemisphere}} = \frac{\pi a^4}{4}$$

Product of Divergences

- Example $\vec{\nabla} \cdot (\vec{r} f(r))$ where $r = |\vec{r}|$ and $f(r)$ is a scalar function

$$= \sum_{\alpha=1}^{\alpha=3} \frac{\partial}{\partial x_\alpha} [f(r) x_\alpha] = \sum_{\alpha=1}^{3} \left[f \frac{\partial x_\alpha}{\partial x_\alpha} + x_\alpha \frac{\partial f}{\partial x_\alpha} \right]$$

$$x_\alpha = (x_1, x_2, x_3) \text{ or } (x, y, z)$$

free index

$$= 3f + \textstyle\sum_1^3 x_\alpha \frac{\partial f}{\partial x_\alpha} = 3f + \vec{r} \cdot \vec{\nabla} f$$

$$= 3f + \vec{r} \cdot \left[f' \frac{\vec{r}}{r} \right] = 3f + r f'$$

Writing all these summation signs gets very tedious! (see ***summation convention*** in Lecture 14).

$$\frac{\partial x_i}{\partial x_j} = \delta_{ij}$$

$$= 1 \text{ if } i = j$$
$$= 0 \text{ if } i \neq j$$

Kronecker Delta: in 3D space, a special set of 9 numbers also known as the ***unit matrix***