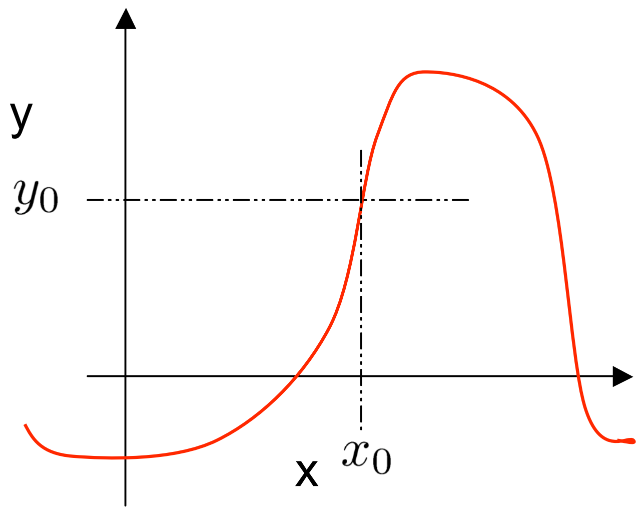
CP3 Revision: Calculus

- Basic techniques of differentiation and integration are assumed
- **Taylor/MacLaurin series** (functions of 1 variable)



- Situation where we want to get value of *y* near a particular point $x = x_0$

Constant Linear Parabolic

$$y(x) \approx a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \ldots$$

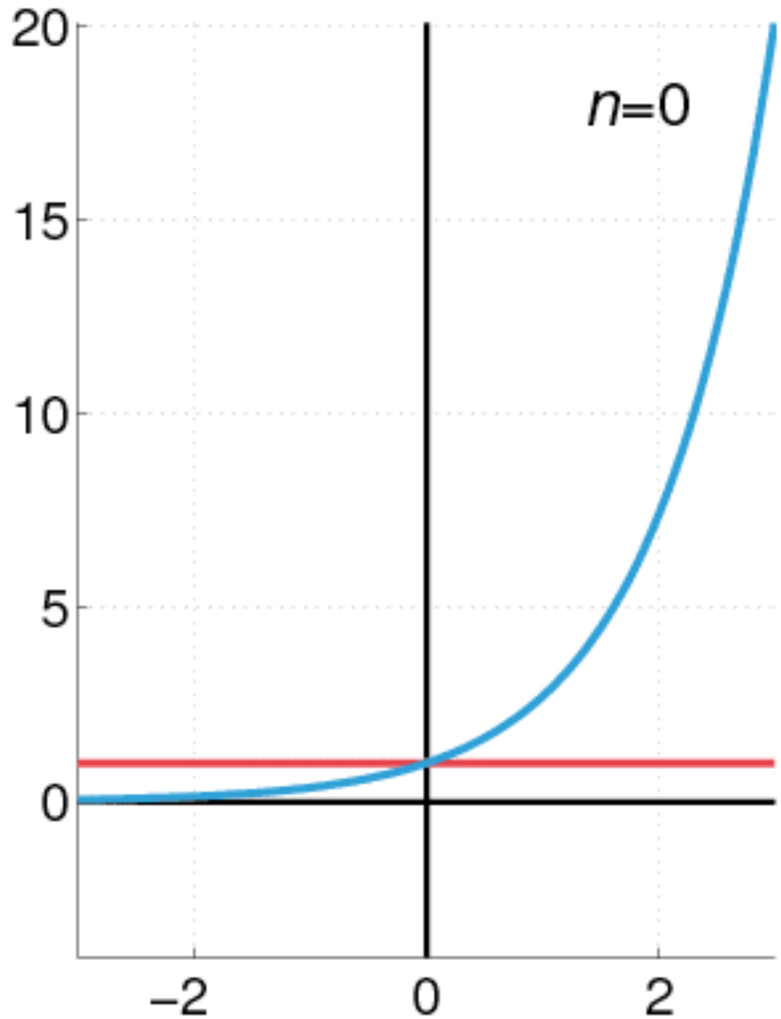
- Value of *n*th coefficient got by substituting $x = x_0$ after differentiating *n* times

$$y(x) = \sum_{n=0}^{\infty} \frac{y^{(n)}(x = x_0)}{n!}(x - x_0)^n$$

is the Taylor series: special case when $x_0 = 0$ is the MacLaurin series

Substituting $\Delta = x - x_0$ gives $y(x_0 + \Delta) = y(x_0) + \Delta y'(x_0) + \frac{\Delta^2}{2!} y''(x_0) + \ldots$

MacLaurin Expansion for e^x



- Note even a tiny number of terms does well near expansion point (*x*=0)
- Note recover complete function at all *x* with infinite number of terms
- powers of *x* just one **basis set** for expanding functions (c.f. Fourier series etc)

Example: determining a limit

- Determine the limit as *x* tends to zero of

$$\frac{(\sin^3 x - x^3)(e^x - 1)(1 + \tan x)}{x^6(1 - \tan x)}$$

using Binomial Expansion

$$\sin^3 x = [x - \tfrac{x^3}{3!} + O(x^6)]^3 \approx x^3[1 - \tfrac{x^2}{3!} + O(x^5)]^3 \approx x^3(1 - \tfrac{3x^2}{3!} + O(x^4))$$

$$\sin x = \cos x(a_0 + a_1 x + a_2 x^2 + \ldots) \rightarrow \tan x = x + O(x^3)$$

Note. Odd function therefore only odd powers of *x*

- So, limit becomes

$$\frac{(x^3 - \frac{x^5}{2} - x^3)(1 + x - 1)(1 + x)}{x^6(1 - x)} = -\frac{1}{2}$$

- Certainly an easier method than L'Hospital's Rule in this case!

Partial Derivatives

- In 2D, for function *h*(*x*,*y*), in the calculus limit: d*x*, d*y* tending to zero

$$\mathrm{d}h = \frac{\partial \mathrm{h}}{\partial x}\mathrm{d}x + \frac{\partial \mathrm{h}}{\partial y}\mathrm{d}y \quad (1)$$

$$= \vec{\nabla}\mathrm{h}.\mathrm{d}\vec{l} \quad \text{if} \quad \vec{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) \quad (2)$$

- Can divide (1) by a variety of things to get various important identities
- If *y* constant so d*y*=0 ,divide (1) by d*h* to get reciprocity relation $1 = \left.\frac{\partial h}{\partial x}\right|_y \left.\frac{\partial x}{\partial h}\right|_y$
- If *y* = *y*(*x*) then divide (1) by d*x* to get $\frac{\mathrm{d}h}{\mathrm{d}x} = \frac{\partial h}{\partial x} + \frac{\partial h}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}x}$
- If *h* is constant d*h*=0, and divide (1) by d*x* to get $\left.\frac{\partial h}{\partial y}\right|_x \left.\frac{\partial y}{\partial x}\right|_h \left.\frac{\partial x}{\partial h}\right|_y = -1$
- If both *x*(*t*) and *y*(*t*) depend on variable *t* then divide by d*t* to get $\frac{\mathrm{d}h}{\mathrm{d}t} = \left.\frac{\partial h}{\partial x}\right|_y \frac{\mathrm{d}x}{\mathrm{d}t} + \left.\frac{\partial h}{\partial y}\right|_x \frac{\mathrm{d}y}{\mathrm{d}t}$
- Finally, if *x*(*t*,*v*) and *y*(*t*,*v*), then divide by d*t* for

$$\left.\frac{\partial h}{\partial t}\right|_v = \left.\frac{\partial h}{\partial x}\right|_y \left.\frac{\partial x}{\partial t}\right|_v + \left.\frac{\partial h}{\partial y}\right|_x \left.\frac{\partial y}{\partial t}\right|_v$$

"Chain Rule" in 2D

Example: change of variables in PDEs

$$\frac{\partial^2 \psi}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 \psi}{\partial t^2}$$

Wave Equation.

Change variables to:

$$\theta = x + ct \quad \phi = x - ct$$

Chain rule gives:

$$\frac{\partial}{\partial x} = \frac{\partial \theta}{\partial x}\frac{\partial}{\partial \theta} + \frac{\partial \phi}{\partial x}\frac{\partial}{\partial \phi} = \frac{\partial}{\partial \theta} + \frac{\partial}{\partial \phi}$$

$$\frac{\partial}{\partial t} = \frac{\partial \theta}{\partial t}\frac{\partial}{\partial \theta} + \frac{\partial \phi}{\partial t}\frac{\partial}{\partial \phi} = c\frac{\partial}{\partial \theta} - c\frac{\partial}{\partial \phi}$$

$$\Rightarrow \left[\frac{\partial}{\partial \theta} + \frac{\partial}{\partial \phi}\right]\left[\frac{\partial}{\partial \theta} + \frac{\partial}{\partial \phi}\right]\psi = \frac{1}{c^2}\left[c\frac{\partial}{\partial \theta} - c\frac{\partial}{\partial \phi}\right]\left[c\frac{\partial}{\partial \theta} - c\frac{\partial}{\partial \phi}\right]\psi$$

$$\Rightarrow 4\frac{\partial^2 \psi}{\partial \theta \partial \phi} = 0 \Rightarrow \psi = f(\theta) + g(\phi) = f(x + ct) + g(x - ct)$$

General solution of 2nd order PDE has two **arbitrary functions** in its general solution: here, waves travelling to left and right with speed c

Example: error analysis

$$g = \frac{4\pi^2 l}{T^2} \Rightarrow \Delta g \approx \frac{\partial g}{\partial l}\mathrm{d}l + \frac{\partial g}{\partial T}\mathrm{d}T$$

Take natural logs and then differentiate

$$\Rightarrow \frac{\Delta g}{g} = \frac{\Delta l}{l} - 2\frac{\Delta T}{T}$$

Square, and on averaging many measurements, this term goes to zero as ‘errors in *l*’ uncorrelated with ‘errors in *T*’

$$\left[\frac{\Delta g}{g}\right]^2 = \left[\frac{\Delta l}{l}\right]^2 + \left[\frac{2\Delta T}{T}\right]^2 - \left(4\frac{\Delta l}{l}\frac{\Delta T}{T}\right)$$

So, variance in *g* is variance in *l* + 4-times the variance in *T*

Integrability Condition

General differential

$$\mathrm{d}\phi = P(x,y)\mathrm{d}x + Q(x,y)\mathrm{d}y$$

is integrable if

$$P = \frac{\partial \phi}{\partial x}; \qquad Q = \frac{\partial \phi}{\partial y}$$

partially differentiate w.r.t. y

partially differentiate w.r.t. x

$$\frac{\partial P}{\partial y} = \frac{\partial^2 \phi}{\partial y \partial x} \qquad \frac{\partial Q}{\partial x} = \frac{\partial^2 \phi}{\partial x \partial y}$$

equal for all well-behaved functions

Or integrability condition $$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

In previous example $P = -y \rightarrow \frac{\partial P}{\partial y} = -1 \quad Q = x \rightarrow \frac{\partial Q}{\partial x} = 1$

Since these are NOT the same, ***not integrable***

Non Conservative Fields

INTEGRATING FACTOR

Turns a ***non-conservative*** vector field into a ***conservative*** vector field.

Example $\vec{A}.\mathrm{d}\vec{l} \text{ with } \vec{A} = (-y, x) \text{ and } \mathrm{d}\vec{l} = (\mathrm{d}x, \mathrm{d}y)$

$\bar{\mathrm{d}}\psi = \vec{A}.\mathrm{d}\vec{l} = -y\mathrm{d}x + x\mathrm{d}y$ is ***inexact*** because if it were ***exact***

$\mathrm{d}\psi = \frac{\partial\psi}{\partial x}.\mathrm{d}x + \frac{\partial\psi}{\partial y}.\mathrm{d}y$ and hence $-y = \frac{\partial\psi}{\partial x} \xrightarrow{\int} \psi = -yx + C(y)$

$x = \frac{\partial\psi}{\partial y} \xrightarrow{\int} \psi = xy + D(x)$

These equations cannot be made consistent for any ***arbitrary functions*** C and D.

Example Integrating Factor

- often, inexact differentials can be made exact with an integrating factor
- Example

$$\frac{1}{x^2+y^2}\text{đ}\psi = \frac{-y}{(x^2+y^2)}\mathrm{d}x + \frac{x}{(x^2+y^2)}\mathrm{d}y$$

- Now

$$\frac{\partial P}{\partial y} = \frac{-(x^2+y^2)+2y^2}{(x^2+y^2)^2}$$

$$\frac{\partial Q}{\partial x} = \frac{(x^2+y^2)-2x^2}{(x^2+y^2)^2}$$

are now equal

$$\mathrm{d}\phi = \frac{\text{đ}\psi}{(x^2+y^2)}$$ defines a potential, or ***state***, function $\phi(x, y)$

Taylor Expansion in 2D

$$\Delta f = f(x,y) - f(x_0,y_0) =$$

$$(x - x_0)\frac{\partial f}{\partial x}\Big|_{(x_0,y_0)} + (y - y_0)\frac{\partial f}{\partial y}\Big|_{(x_0,y_0)} +$$

Approximates surface by tilted flat plane, c.f. total differential

$$\tfrac{1}{2!}\begin{pmatrix} x - x_0 & y - y_0 \end{pmatrix}\begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix}\begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix} + \ldots$$

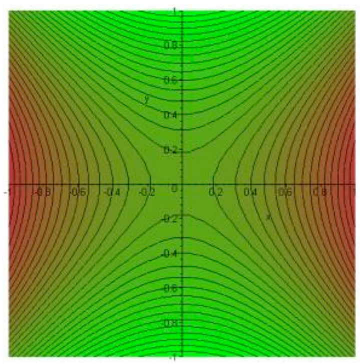
adds a parabaloid term

Higher-order terms add accuracy as in 1D case

Extrema & Saddle Points

- Points where $f_x = f_y = 0$, and WLOG choose origin at each of these points

$$\Delta f \approx \frac{1}{2!}\begin{pmatrix} x & y \end{pmatrix}\begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \dots$$



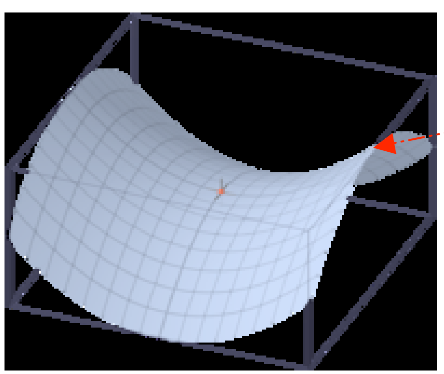
Rotating co-ordinate axes (x,y) to (X,Y)

$$\Delta f \approx \frac{1}{2}(\lambda_1 X^2 + \lambda_2 Y^2)$$

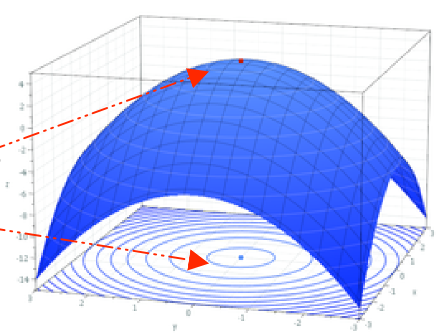
Where λ_1 and λ_2 are the **eigenvalues** of this matrix; recall **determinant D** = $\lambda_1 \lambda_2$ = $f_{xx} f_{yy} - f_{xy}^2$

Two cases

(i) λ_1, λ_2 opposite sign. Contours of Δf are hyperbolae. Saddle point. D < 0.



(ii) λ_1, λ_2 same sign. Contours of Δf are ellipses. Extremum. D > 0.



2D Jacobian

- For a continuous 1-to-1 transformation from (*x,y*) to (*u,v*)
- Then $x = x(u,v) \text{ and } y = y(u,v)$

$$\int\int_R f(x,y)\mathrm{d}x\mathrm{d}y = \int\int_{R'} f(x(u,v),y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| \mathrm{d}u\mathrm{d}v$$

- Where Region (in the *xy* plane) maps onto region R in the *uv* plane R'

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

2D Jacobian

maps areas dx*dy* to areas d*u*d*v*

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \left|\frac{\partial(u,v)}{\partial(x,y)}\right|^{-1}$$

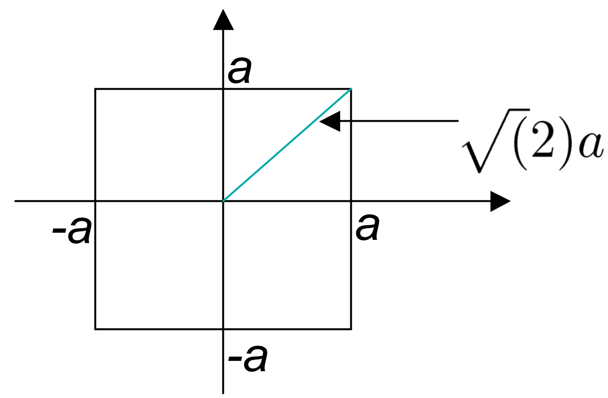
- Hereafter call such terms x_u

$$= \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = x_u y_v - x_v y_u$$

An Important 2D Example

- Evaluate

$$I = \int_{-\infty}^{\infty} e^{-x^2} \mathrm{d}x$$



- First consider

$$I_a^2 = \int_{-a}^{a} e^{-x^2} \mathrm{d}x \int_{-a}^{a} e^{-y^2} \mathrm{d}y$$

$$I_a^2 = \int_{-a}^{a} \int_{-a}^{a} e^{-(x^2+y^2)} \mathrm{d}y \mathrm{d}x$$

- Put $x = r\cos\phi$ and $y = r\sin\phi$

$$\frac{\partial(x,y)}{\partial(r,\phi)} = \begin{vmatrix} \cos\phi & -r\cos\phi \\ \sin\phi & r\cos\phi \end{vmatrix} = r$$

$$\int_0^a \int_0^{2\pi} r e^{-r^2} \mathrm{d}r \mathrm{d}\phi < I_a^2 < \int_0^{\sqrt{2}a} \int_0^{2\pi} r e^{-r^2} \mathrm{d}r \mathrm{d}\phi$$

- $\pi(1 - e^{-a^2}) < I_a^2 < \pi(1 - e^{-2a^2})$ as $a \longrightarrow \infty \Rightarrow I_a = \sqrt{\pi}$

3D Jacobian

$$x = x(u, v, w), y = y(u, v, w), z = z(u, v, w)$$

- maps volumes (consisting of small cubes of volume $\mathrm{d}x\mathrm{d}y\mathrm{d}z$
- ....to small cubes of volume $\mathrm{d}u\mathrm{d}v\mathrm{d}w$

$$\int\int\int_V f(x, y, z)\mathrm{d}x\mathrm{d}y\mathrm{d}z = \int\int\int_{V'} F(u, v, w)\left|\frac{\partial(x, y, z)}{\partial(u, v, w)}\right|\mathrm{d}u\mathrm{d}v\mathrm{d}w$$

- Where

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{vmatrix}$$

3D Example

- Transformation of volume elements between Cartesian and spherical polar coordinate systems (see Lecture 4)

$$
\begin{aligned}
x &= r\sin\theta\cos\phi \\
y &= r\sin\theta\sin\phi \\
z &= r\cos\theta
\end{aligned}
$$

$$
\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}
$$

$$
= r^2\sin\theta
$$

Evaluation of Surface Integrals by Projection

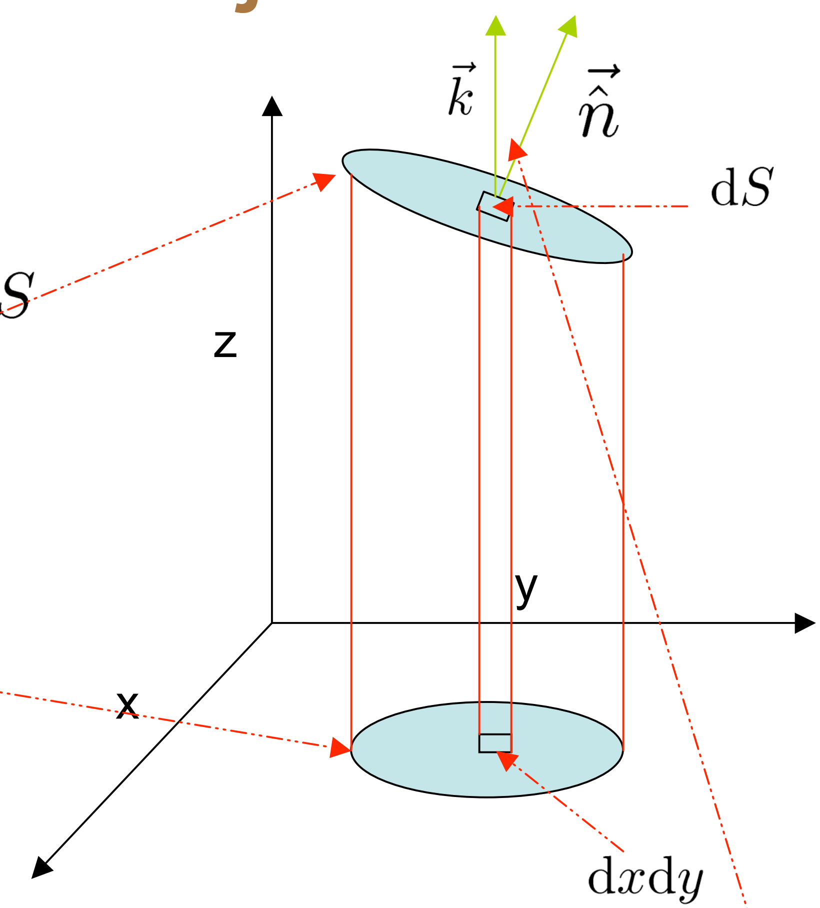
want to calculate

$$\int\int_S \vec{A}.\mathrm{d}\vec{S} = \int\int_S \vec{A}.\vec{\hat{n}}\,\mathrm{d}S$$

$$= \int\int_R \vec{A}.\vec{\hat{n}}\frac{\mathrm{d}x\mathrm{d}y}{|\vec{n}.\vec{k}|}$$

because

$$\mathrm{d}S\cos\theta = \mathrm{d}x\mathrm{d}y$$



$$\vec{\hat{n}}.\vec{k} = |\vec{\hat{n}}||\vec{k}|\cos\theta = \cos\theta$$

Note S need not be planar!

Note also, project onto easiest plane

Example

- Surface Area of some general shape $z = f(x, y)$

$$= \int \int_R \vec{A}.\vec{\hat{n}} \frac{\mathrm{d}x\mathrm{d}y}{|\vec{n}.\vec{k}|} \quad \text{where } \vec{A} = \vec{\hat{n}}$$

$$\vec{\hat{n}} = \frac{(f_x, f_y, -1)}{\sqrt{1+f_x^2+f_y^2}}$$

$$= \int \int_R \sqrt{1 + f_x^2 + f_y^2}\mathrm{d}x\mathrm{d}y$$