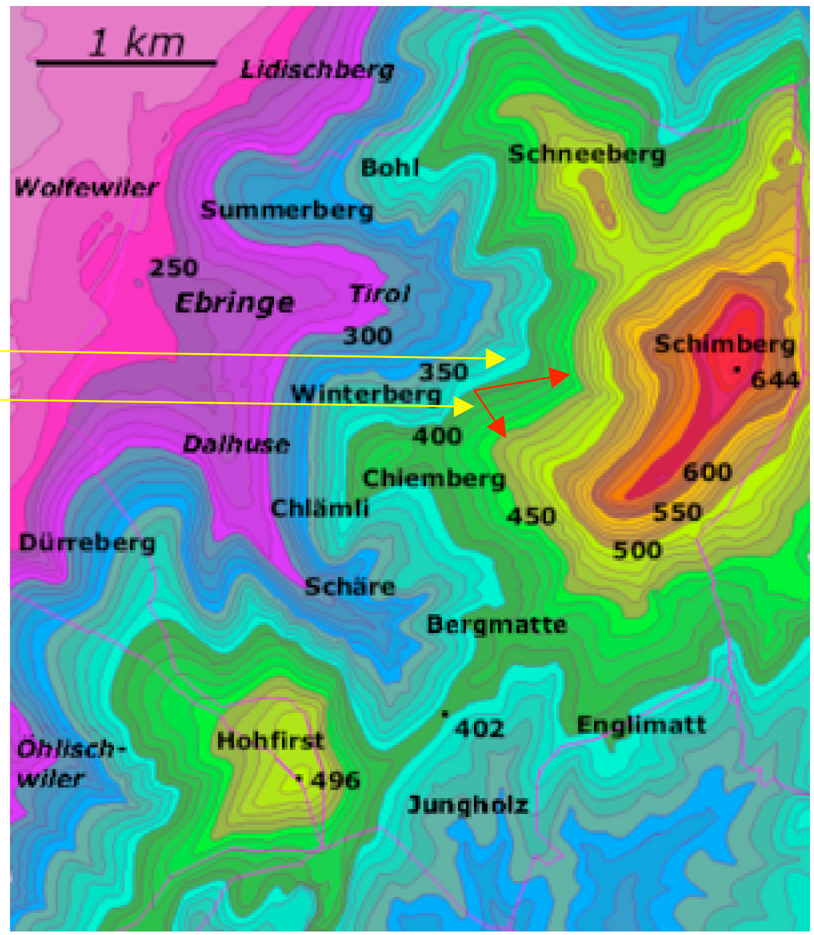
CP3 Revision: Vector Calculus

- Need to extend idea of a gradient (df/dx) to 2D/3D functions
- Example: 2D scalar function h(x,y)
- Need "dh/dl" but dh depends on direction of dl (greatest up hill), define dl_{max} as short distance in this direction
- Define $\vec{\text{grad}}(h) \quad \text{magnitude} = |\frac{\text{d}h}{\text{d}l_{\text{max}}}|$
- Direction, that of steepest slope

$$\begin{aligned} \text{d}h &= \frac{\partial \text{h}}{\partial x}\text{d}x + \frac{\partial \text{h}}{\partial y}\text{d}y \\ &= \vec{\nabla}\text{h}.\vec{\text{d}l} \quad \text{if} \quad \vec{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) \end{aligned}$$



Vectors always perpendicular to contours

So if $\vec{\mathrm{d}l}$ is along a contour line

$$\mathrm{d}h = \vec{\nabla}h.\vec{\mathrm{d}l}_{\mathrm{cont}} = 0 \rightarrow \text{ direction of } \vec{\nabla}h$$

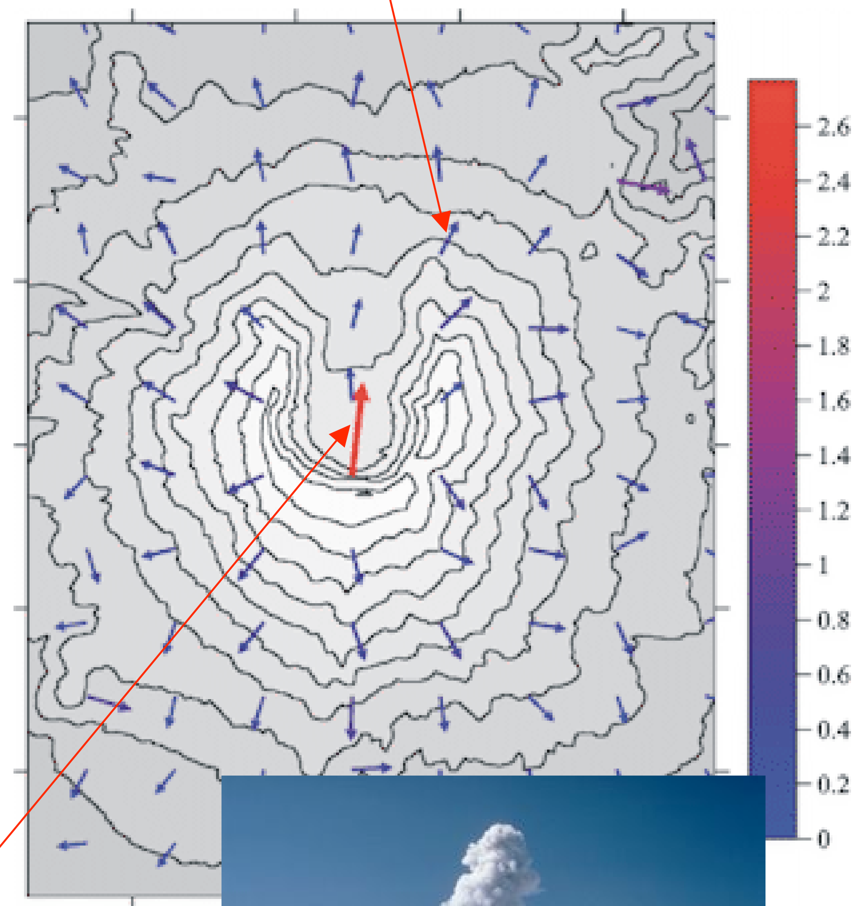
Is perpendicular to contours, ie up lines of steepest slope

And if $\vec{\mathrm{d}l}$ is along this direction

$$\mathrm{d}h = |\vec{\nabla}h|\mathrm{d}l_{\mathrm{max}} \rightarrow |\vec{\nabla}h| = \frac{\mathrm{d}h}{\mathrm{d}l_{\mathrm{max}}}$$

$$\vec{\mathrm{grad}}(h) = \vec{\nabla}h = \left(\frac{\partial \mathrm{h}}{\partial x}, \frac{\partial \mathrm{h}}{\partial y}\right)$$

The vector field shown is of $-\vec{\nabla}h$



Magnitudes of vectors greatest where slope is steepest

Del

$$\vec{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right)$$

Grad

$$\vec{\nabla}\phi = \vec{A}$$

"Vector operator acts on a scalar field to generate a vector field"

Example:

$$\vec{\nabla}(xy + y^2 z + \sin z) = (y, x + 2yz, y^2 + \cos z)$$

Grad Example: Tangent Planes

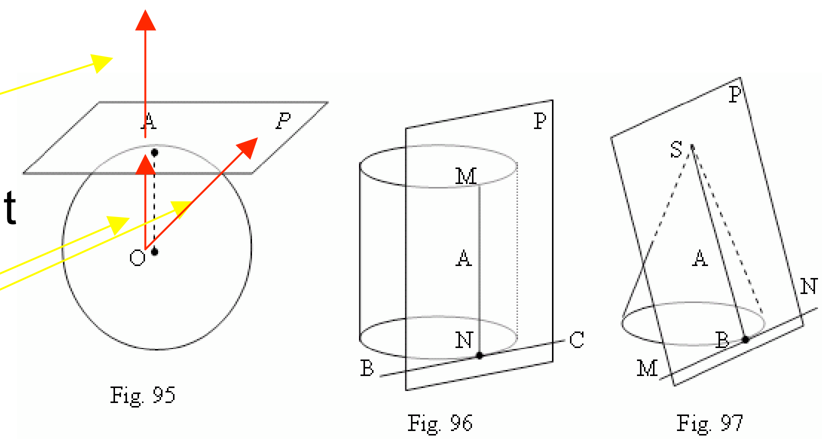
- Since $\vec{\nabla} f$ is perpendicular to contours, it locally defines direction of normal to surface

$$f(x, y, z) = A$$

- Defines a family of surfaces (for different values of A)
- $\vec{\nabla} f$ defines normals to these surfaces
- At a specific point $\vec{r_0} = (x_0, y_0, z_0)$

  tangent plane has equation

$$\vec{r}.\vec{\nabla} f|_{(x_0, y_0, z_0)} = \vec{r_0}.\vec{\nabla} f|_{(x_0, y_0, z_0)}$$



Conservative Fields

In a Conservative Vector Field $\vec{A}$

$$\int_1^2 \vec{A}.\vec{\mathrm{d}l} \text{ is independent of path } \equiv \text{vector field } \vec{A} = \vec{\nabla}\phi \text{ for scalar potential } \phi$$

$$\mathrm{d}\phi = \frac{\partial \phi}{\partial x}\mathrm{d}x + \frac{\partial \phi}{\partial y}\mathrm{d}y + \frac{\partial \phi}{\partial z}\mathrm{d}z = \vec{\nabla}\phi.\vec{\mathrm{d}l}$$

$$\int_1^2 \vec{A}.\vec{\mathrm{d}l} = \mathrm{d}\phi = \phi_2 - \phi_1$$

Which gives an easy way of evaluating line integrals: regardless of path, it is difference of potentials at points 1 and 2.

$$\oint \vec{A}.\vec{\mathrm{d}l} = 0$$

Obvious provided potential is single-valued at the start and end point of the closed loop.

$$\mathrm{div}\vec{A} = \vec{\nabla}.\vec{A}$$

Div

$$\mathrm{div}\vec{A} \equiv \lim_{\mathrm{d}V \to 0} \frac{\oint \vec{A} \cdot \mathrm{d}\vec{S}}{\mathrm{d}V}$$

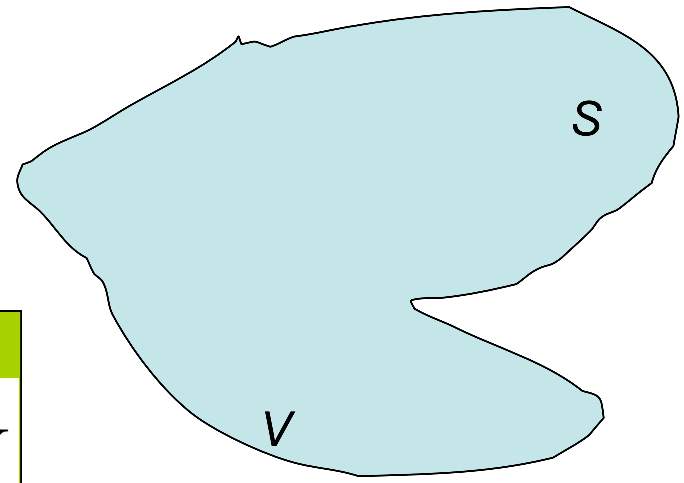
$$\vec{\nabla} \cdot \vec{A} = \psi$$

“Vector operator acts on a vector field to generate a scalar field”

Example

$$\vec{\nabla} \cdot (xy, y^2 z, \sin z) = y + 2yz + \cos z$$

Divergence Theorem



$$\Rightarrow \oint\!\!\oint_S \vec{A} \cdot \mathrm{d}\vec{S} = \underbrace{\int\!\!\int\!\!\int_V \vec{\nabla} \cdot \vec{A}\, \mathrm{d}V}$$

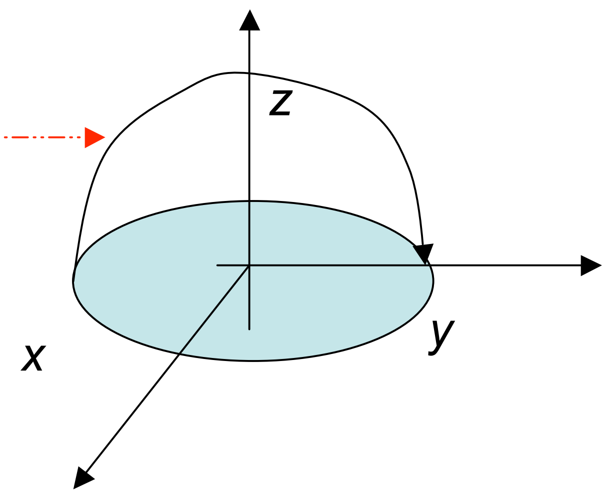
Oover ***closed*** outer surface *S* enclosing *V*

Divergence Theorem as aid to doing complicated surface integrals

- Example $\int\int_S \vec{F}.\mathrm{d}\vec{S}$ over

open hemisphere $x^2 + y^2 + z^2 = a^2, z \geq 0$

$$\vec{F} = (y - x, x^2 z, z + x^2)$$



- Evaluate Directly $\mathrm{d}\vec{S} = \underbrace{\frac{1}{a}(x, y, z)}_{\hat{\vec{n}}}\ \mathrm{d}S$

on surface $x = a \sin\theta \cos\phi$

$y = a \sin\theta \sin\phi$

$z = a \cos\theta$

$\mathrm{d}S = a^2 \sin\theta \mathrm{d}\theta \mathrm{d}\phi$

- Tedious integral over θ and φ (exercise for student!) gives

$$\frac{\pi a^4}{4}$$

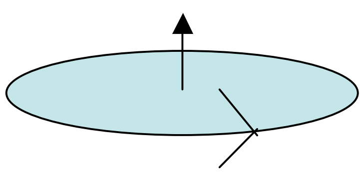
Using the Divergence Theorem

$$\vec{\nabla}.\vec{F} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right).(y - x, x^2 z, z + x^2) = -1 + 0 + 1 = 0$$

- So integral depends only on ***rim***: so do easy integral over circle

$$x^2 + y^2 = a^2, z = 0$$

$$\int\int (y - x, 0, x^2).(0, 0, 1)\mathrm{d}x\,\mathrm{d}y$$



$$= \int_0^a \int_0^{2\pi} r^2 \cos^2\phi\, r \mathrm{d}r \mathrm{d}\phi = \frac{a^4}{4}\int_0^{2\pi} \cos^2\phi\,\mathrm{d}\phi$$

$$= \frac{\pi a^4}{4}$$

- as always beware of signs

$$\int\int_{\mathrm{hemisphere}} - \frac{\pi a^4}{4} = 0 \Rightarrow \int\int_{\mathrm{hemisphere}} = \frac{\pi a^4}{4}$$

Curl

$$\text{curl}\vec{A} \quad \lim_{\text{area}\to 0} \frac{\oint \vec{A}.\mathrm{d}\vec{l}}{\text{area}}$$

Magnitude

Direction: normal to plane which maximises the line integral.

Can evaluate 3 components by taking areas with normals in *xyz* directions

$$\vec{\nabla} \times \vec{A} = \vec{B}$$

"Vector operator acts on a vector field to generate a vector field"

Example

$$\vec{\nabla} \times (xy, y^2 z, \sin z) = (-y^2, 0, -x)$$

Key Equations

$$\oint \vec{A}.\mathrm{d}\vec{l} \equiv (\mathrm{curl}\vec{A}).\mathrm{d}\vec{S}$$

$$\mathrm{curl}\vec{A} = \left[\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}, \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}, \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right]$$

$$\mathrm{curl}\vec{A} = \vec{\nabla} \times \vec{A} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

Simple Examples

$$\vec{\nabla} \times \vec{r} = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}) \times (x, y, z)$$

$$= (\frac{\partial z}{\partial y} - \frac{\partial y}{\partial z}, \frac{\partial x}{\partial z} - \frac{\partial z}{\partial x}, \frac{\partial y}{\partial x} - \frac{\partial x}{\partial y}) = (0, 0, 0)$$

Radial fields have zero curl

$$\vec{\nabla} \times (-y, x, z)$$

$$= (\frac{\partial z}{\partial y} - \frac{\partial x}{\partial z}, \frac{\partial(-y)}{\partial z} - \frac{\partial z}{\partial x}, \frac{\partial x}{\partial x} - \frac{\partial(-y)}{\partial y}) = (0, 0, 2)$$

Rotating fields have curl in direction of rotation

Example

$$\vec{A} = (xy^2 + z, x^2y + 2, x) \Rightarrow \vec{\nabla} \times \vec{A} = (0, 1 - 1, 2xy - 2xy) = \vec{0}$$

- Irrotational and conservative are synonymous because

$$\oint \vec{A} \cdot \mathrm{d}\vec{l} = 0 \Rightarrow \vec{A} = \vec{\nabla}\phi \overset{\text{by Stokes Theorem}}{\Rightarrow} \mathrm{curl}\vec{A} = \vec{0} \Rightarrow \mathrm{curl}(\mathrm{grad}\phi) = \vec{0}$$

- So this is a ***conservative*** field, so we should be able to find a ***potential*** ϕ

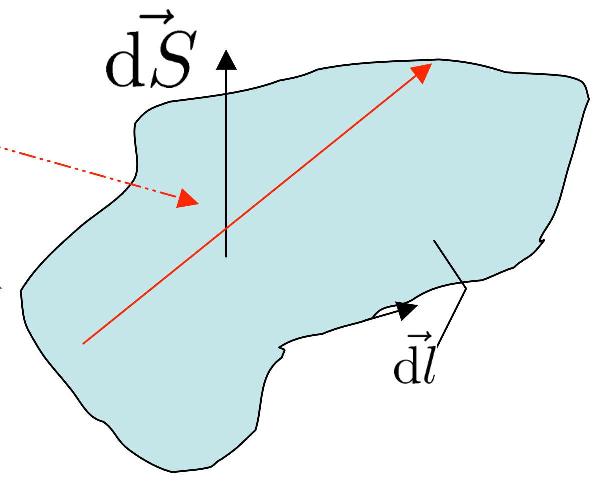
$$\frac{\partial \phi}{\partial x} = xy^2 + z \Rightarrow \phi = \frac{1}{2}x^2y^2 + zx + f(y, z)$$
$$\frac{\partial \phi}{\partial y} = x^2y + 2 \Rightarrow \phi = \frac{1}{2}x^2y^2 + 2y + g(x, z)$$
$$\frac{\partial \phi}{\partial z} = x \Rightarrow \phi = xz + h(x, y)$$

- All can be made consistent if

$$\phi = \frac{1}{2}x^2y^2 + zx + 2y + k \text{ where } k \text{ is a constant}$$

Stokes Theorem

- Consider a surface *S*, embedded in a vector field $\vec{A}$
- Assume it is bounded by a rim (not necessarily planar)

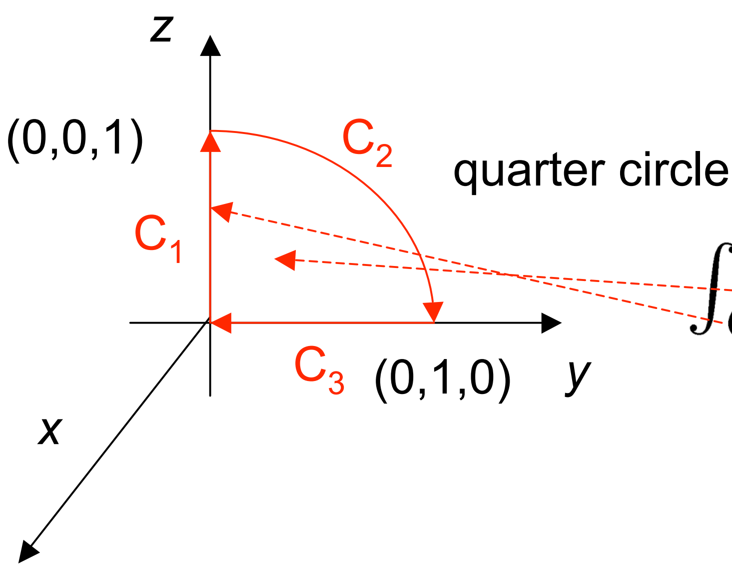


$$\oint \vec{A}.\mathrm{d}\vec{l} = \int\int_S \mathrm{d}\vec{S}.(\vec{\nabla} \times \vec{A})$$

OUTER RIM SURFACE INTEGRAL OVER ***ANY***

SURFACE WHICH SPANS RIM

Example



$$\vec{F}(x,y,z) = (y,z,x)$$

$$\int_C \vec{F}\cdot \mathrm{d}\vec{l} = \int\int_S (\vec{\nabla}\times\vec{F})\cdot \mathrm{d}\vec{S}$$

$$\vec{\nabla}\times\vec{F} = (-1,-1,-1)$$

$$\mathrm{d}\vec{S} = (-1,0,0)\mathrm{d}y\mathrm{d}z$$

$$\Rightarrow \int_C \vec{F}\cdot \mathrm{d}\vec{l} = \frac{\pi}{4}$$

- Check via direct integration $\int_C = \int_{C_1} + \int_{C_2} + \int_{C_3} = 0 + \frac{\pi}{4} + 0$

2nd-order Vector Operators

	2nd	grad	div	curl
1st	grad	$\times$	OK	$\times$
	div	∇^2	$\times$	0
	curl	$\vec{0}$	$\times$	OK

Lecture 11

$\times$ meaningless

$$\nabla^2 = \vec{\nabla}.\vec{\nabla} \equiv \mathbf{DEL-SQUARED} \equiv \mathbf{LAPLACIAN}$$

Laplace's Equation $\nabla^2\phi = 0$ is one of the most important in physics

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \vec{\nabla}(\vec{\nabla}.\vec{A}) - \vec{\nabla}^2\vec{A}$$